1960

On the determination of the dimensions of the earth ellipsoid.

Bergford, Paul Martin

Monterey, California. Naval Postgraduate School

ON THE DETERMINATION OF THE DIMENSIONS OF THE EARTH ELLIPSOID

PAUL MARTIN BERGFORD

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DIMENSIONS OF THE EARTH ELLIPSOID

A Thesis

Presented in Partial Fulfillment of the Requirements
for the Degree Master of Science

by

Paul Martin Bergford, B.E.

The Ohio State University
1960

Preface

It has been noted, in the course of study of geodesy, that there is a sparsity of material devoted to the mechanics of determining the size and shape of the reference ellipsoid from actual observations. Bare outlines of some methods are given in the few current textbooks available in English, and although some of the older texts, notably Clarke (5) and Crandall (6), do give examples, the methods used are rather ancient and outmoded and in notation systems quite unfamiliar to the present-day student. Hayford's original study of 1909 is the best available (10). Since one of the basic aims of geodesy is to determine the figure of the earth, it seemed like a good idea to pursue the matter, if even yet somewhat superficially, to the point where there would be few questions left concerning the manner of going about the task.

The writer wishes to acknowledge the help and advice provided him in preparation of this thesis, particularly by his adviser, Dr. W.A. Heiskanen, Director of the Institute of Geodesy, Photogrammetry, and Cartography, the Ohio State University. Acknowledgement is also due Dr. U.A. Uotila and Mr. Ivan Mueller, both of the Ohio State University, for certain helpful suggestions; the U.S. Army Map Service and Mr. W.M. Kaula, for data on points in the Finland - South Africa arc; the U.S. Navy, without the instigation of which, the course of study referred to in the opening sentence would not have been possible.

Finally, the writer wishes to express his appreciation to Frauke and Peter Wilson, the former for her fine job of typing the bulk of this thesis on short notice, and the latter for certain editorial assistance to the typist.

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Chapter 1. Introduction

1.1 Preleminary Remarks.

That the most convenient practical reference surface for purposes of geodetic computation is the ellipsoid of revolution has long been recognized. This is emphasized by the fact that the major datums in use in the world today are all based on ellipsoids of revolution. The fact that this shape is a relatively simple regular mathematical surface, and that it closely approximates the mean geoidal form, make its choice quite understandable. The Russians, who have been proponents of a triaxial ellipsoid, do not even consider its use for practical geodetic computations. Izotov (18) states:-

> "The triaxial ellipsoid could not be accepted for triangulation processing, as this would have made the working out of geodetic measurements too complicated".

To dismiss the subject of triaxiality for purposes of this paper, the following is cited from "The Earth and its Gravity Field", by Heiskanen and Vening Meinesz:-

> "It appears that we had better forget the triaxial ellipsoid, particularly because since the earth is in close isostatic equilibrium, it can hardly have the shape of a triaxial ellipsoid." (13) p. 79-80.

The use of geodetic data to determine relatively precisely what size and how much flattened the reference ellipsoid should be, is the subject of this discussion. In this connection, one of the principal purposes of geodesy has been variously stated:-

> "Its theoretical function is to determine the size and shape of the earth ..." (13); "Geodesy is the science which treats of investigations of the form and dimensions of the earth's surface." (17); "The accurate determination of the figure of the earth." (6); "In combination with observations for latitude, longitude, and gravity to assist in determining the size and shape of the earth, ..." (2); "It may be said that the ultimate aim of scientific geodesy is to determine the size and shape of the geoid, ..." (2).

Of course, data collected in geodetic surveys along with the calculations connected therewith, and computations and results of computations of the figure of the earth are mutually interdependent.

Although the problem of determining the best size and shape for the reference ellipsoid has decreased in importance relative to other modern forms of geodetic endeavor, there are still significant efforts being made in this field, notably by the U.S. Army Map Service. This is not to say that the exact dimensions of the earth are not important, but the position has now been reached where further precision given to a reference ellipsoid must be subordinated to the problem of determining the deviations of the actual form of the geoid with respect to the reference surface selected.

This change of emphasis is described somewhat succinctly by Tengstrom:-

> "The principal problem of geodesy is to determine the geocentric coordinates of all points on the earth's surface, a task which was earlier referred to as a determination of the 'size and shape' of the earth." (30);

or by L.G. Simmons:-

> "The ultimate goal of the geodesist is, I suppose, to determine the parameters of an ellipsoid of revolution which best fits the figure of the earth as a whole. But this is not all. He also

is concerned with the details of lack of fit of this ellipsoid to the actual Earth's shape, the geoid." (28).

Most important would be the selection of one reference surface to be used in problems of world-wide extent, such as for requirements of missile warfare, the success of which might very well be determined by accurate knowledge of locations of launch sites and targets. Naturally, the best reference surface obtainable is to be preferred. The establishment of a "World Geodetic System" and conversion of existing systems to it, has for several years been one of several significant projects promoted and encouraged by Dr. Heiskanen (16) (13) p. 299-310. Both the U.S. Air Force and the U.S. Army Map Service are developing world geodetic systems for military purposes. These are eventually to be combined into a single unified Department of Defense World Datum, the contributions to the establishment of which will be made by all U.S. military services. The reference ellipsoid to be used for this will be based on all available information from all parts of the world. (3) p. 77.

Having exposed the near-obsolescent character of the problem of determining the dimensions of the reference ellipsoid, we may now proceed to describe the several variations of its solution.

1.2 General Discussion of Methods.

The standard approach to the problem under consideration has been, from earliest times right to the present day, that contained in the arc method. Variations of this approach have become refined and quite sophisticated, taking into account the earth's isostatic

equilibrium, and in some cases being combined with the gravimetric method. The earliest recorded application of the arc method is told in the now trite story of Eratosthenes' activities in this field.

The arc method assumes that short meridional arcs on the earth's surface are arcs of circles; when two meridian arcs are measured, one at relatively high latitude, and one near the equator, both the equatorial radius and the flattening can be computed. The more such arcs used, combined in a least squares solution, the better accuracy is obtained.

A more convenient adaptation of the arc method has been used since early in the present century. This uses a combination of geodetic and astronomic data in the form of astro-geodetic deflections of the vertical. Hayford used this method in developing what is now known as the International Ellipsoid (10). He called the procedure he used the "area method", because the data was taken from many points in a large network of triangulation. He made several solutions, the deflections of the vertical in each being reduced to the geoid (co-geoid) according to the theory of isostasy using different depths of compensation. He chose the solution in which the sum of the squares of the residual deflections after adjustment was the least. Thus a "by-product" of his work was a solution of the mean depth of compensation for the earth's crust.

The application of the gravimetric method to the problem under consideration is really a combination of the arc method, modified by the introduction of gravimetrically determined deflections of the

vertical, to obtain the earth's equatorial radius, and the determination of the earth's flattening from gravity anomalies by use of Clairaut's formula.

The latest, and of course the most modern method used to determine the earth's shape, is the artificial earth satellite method. By measuring the perturbations in the satellite's orbit around the earth, caused by deviations from the spherical in the earth's shape, the flattening is obtained. This method is so new in actual application, that an evaluation of results thus far obtained can only be the subject of conjecture.

One other method, similar to the deflection of the vertical application of the arc method, involves the use of the geoid undulations, obtained both gravimetrically and astro-geodetically. The solution is obtained in reducing the sum of the squares of the differences between the two at many points to a minimum.

Chapter 2. Dimensions from Two Arcs

2.1 Meridional Arcs.

In this analysis, as in all subsequent ones in this paper, the reference surface is assumed to be an ellipsoid of revolution. The mathematical properties of an ellipse form the basis for expressions referring to the meridian of such a surface.

The measurement of two arcs in meridian, one near the pole and the other near the equator, provides the data for this, the simplest means whereby the earth dimensions may be obtained. The arcs, which may be on different meridians, are measured, and the difference in latitudes between extremities for each is computed, the latitude of one end point for each arc having been determined astronomically.

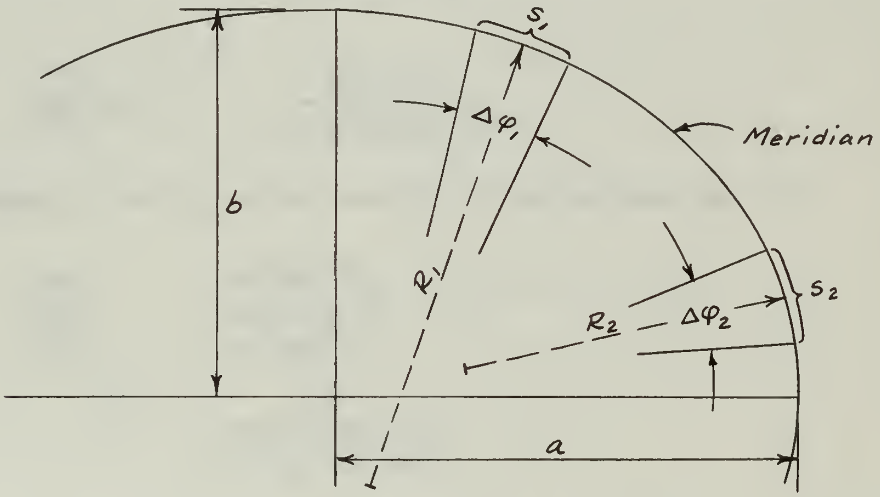

Fig. 1

In fig. 1, φ_1 and φ_2 are the mid-latitudes, and s_1 and s_2 the measured lengths of the two arcs. Assuming s_1 and s_2 to be arcs of circles, we can write:-

$$(2\text{-}1) \qquad R_1 = \frac{s_1}{\Delta\varphi_1} \csc 1''$$

$$R_2 = \frac{s_2}{\Delta\varphi_2} \csc 1''$$

$\Delta\varphi_1$ and $\Delta\varphi_2$ being expressed in seconds.

The expressions for the radii of curvature of the elliptical meridian at the respective latitudes are:

$$(2\text{-}2) \qquad R_1 = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi_1)^{3/2}}$$

$$R_2 = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi_2)^{3/2}}$$

Equating the two expressions (2-1) and (2-2), we obtain:

$$(2\text{-}3) \qquad \frac{s_1}{\Delta\varphi_1} \csc 1'' = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi_1)^{3/2}}$$

$$\frac{s_2}{\Delta\varphi_2} \csc 1'' = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi_2)^{3/2}}$$

Dividing R_1 by R_2 in the form (2-3) and solving for e^2:

$$(2\text{-}4) \qquad \frac{s_1\Delta\varphi_2}{s_2\Delta\varphi_1} = \frac{1-e^2\sin^2\varphi_2}{1-e^2\sin^2\varphi_1} = q^2$$

$$e^2 = \frac{1-q^2}{\sin^2\varphi_2 - q^2\sin^2\varphi_1}$$

Then, the value of <u>a</u> can be found from (2-3):

$$(2\text{-}5) \qquad a = \frac{s_1(1-e^2\sin^2\varphi_1)^{3/2}}{\Delta\varphi_1(1-e^2)\sin 1''} = \frac{s_2(1-e^2\sin^2\varphi_2)^{3/2}}{\Delta\varphi_2(1-e^2)\sin 1''}$$

Now all that remains is to determine the flattening, which is simply:

$$f = \frac{a-b}{a} = 1 - \frac{b}{a} = 1 - \sqrt{1-e^2} \tag{2-6}$$

2.2 Oblique Arcs.

In the early efforts of measuring the earth's figure relatively short arcs were measured, almost always in meridian. In later determinations, existing triangulation data was put into use, and consequently successive points used were not always on the same meridian. Such geodetically computed arcs, chosen so as to be inclined to the meridian at as small an angle as possible, could be reduced to the meridian by appropriate formulae or from tables based on the used ellipsoid. Thus armed with the same data as in section 2.1 a similar problem can be solved.

When the length of the oblique arc is measured, along with its azimuth and mid-latitude, its meridian distance is given by:

$$\begin{aligned} \Delta\varphi R \sin 1'' &= - s \cos A - \frac{1}{2N} S^2 \sin^2 A \tan \varphi \\ &+ \frac{1}{6N^2} S^3 \sin^2 A \cos A \,(1+3\tan^2\varphi) \end{aligned} \tag{2-7}$$

where N is the radius of curvature of the prime vertical and A is the azimuth, reckoned positive clockwise from south.

Using (2-7) a chain of triangles can be projected to the meridian for the whole length of the chain. Given geodetic latitudes, however, use can be made of specially prepared table, such as (34), (35), or (36), which have meridian distance tabulated against latitude for the ellipsoid involved.

The above method of determining the earth's dimensions (Sections 2.1 and 2.2), though it represents the basis for other methods, is mainly of historical and mathematical interest, since no longer are serious scientific computations carried out in such simple terms. References for the presentation given in these two sections are (6), (17), and (23).

Chapter 3. Dimensions from Several Arcs.

3.1 General.

When geodesists of the 18th Century first began to compare the results of their arc measuring efforts, they found such large discrepancies that they began to speculate about the accuracy of their measurements and about the correctness of the assumption that the earth's true shape is an ellipsoid of revolution. Of course the accuracy of surveys of those early days was not what it is today. Nor could one hope to obtain the same results from arcs, taken two by two in different parts of the world having varying kinds of topography, measured by the crude methods of the day. Not only was the measurement of geodetic triangles so rude that spherical excess remained undetected, but the process of adjustment by the method of least squares was unknown (23).

Obviously, not wanting to discard such a convenient reference surface as an ellipsoid, which after all departs from the geoid surface by probably less than 100 m. at the maximum, and from the equilibrium figure by even less (of the order of 2 m. at $\varphi = 0, 45$, and 90, (13) p. 59), the best thing to do would be to combine the results of several arc-measurements in such a way as to determine the mean values for the earth as a whole.

3.2 Laplace's Method.

Toward the end of the 18th Century, Laplace published the results of his attempt at combining the data from nine different sources,

extending from Lapland to Cape of Good Hope in latitude. His method involved the data corresponding to an arc of one degree of latitude for each of nine different meridians.

Again considering these one degree arcs as arcs of circles and using R, the radius of curvature of the midpoint of the arc, and φ, the mid-latitude, we have:-

$$d = \frac{2\pi R}{360} = \frac{\pi a\,(1-e^2)}{180\,(1-e^2\sin^2\varphi)^{3/2}} \tag{3-1}$$

Developing this in series, we obtain:-

$$d = \frac{\pi a(1-e^2)}{180}\left(1 + \frac{3}{2}e^2\sin^2\varphi + \frac{15}{8}e^4\sin^4\varphi + \ldots\right) \tag{3-2}$$

which can be written in the following form:-

$$d = M + N\sin^2\varphi + P\sin^4\varphi + \ldots \tag{3-3}$$

in which $M = \frac{\pi a\,(1-e^2)}{180}$

$$N = \frac{3}{2}e^2 M$$

$$P = \frac{15}{8}e^4 M, \ldots \text{ etc.}$$

Powers of $\underline{e}$ higher than the square are neglected.

In order to combine the data for each degree-measurement in the manner yielding the least discrepancies, the observation equations for each arc were of the form:-

$$d_1 - M - N\sin^2\varphi = v_1 \tag{3-4}$$

$$d_2 - M - N\sin^2\varphi = v_2 \qquad \text{etc.}$$

in which v_1, v_2, etc., are the residual errors. Since the method of least squares was unknown at the time, Laplace combined the error

equations in such a way that the absolute values of the residual errors would be a minimum, and the algebraic sum equal to zero. Then solving simultaneous equations for M and N, the values of $\underline{e}^2$ and $\underline{a}$ are found. Of course, this problem is easily adapted to least squares solution.

The results of Laplace's computation are:-

$$M = 11\ 061.2 \text{ meters}$$

$$N = 1\ 194.9 \text{ meters}$$

$$e^2 = 0.007262$$

$$a = 6\ 383\ 600 \text{ meters}$$

$$f = 1/278$$

The error, or difference between a measured degree and one computed from the result of the crude adjustment of the nine arcs, is of the order of 268 meters. From this, Laplace concluded that the earth deviates materially from an elliptical figure (23).

3.3 By Correction of Latitude.

A refinement of Laplace's method, in use by the end of the 19th Century, is here described (6), (17). This also involves the formation of observation equations between observed latitudes and either directly measured meridian arcs or oblique arcs projected to the meridian.

For short arcs the relation $s = \Delta\varphi\, R \sin 1''$ can be used, but for longer arcs a correction is required. Crandall (6) states that the simple relation given is good for a $\Delta\varphi$ of "several degrees" because of the large probable error of latitude determination, some $0\overset{''}{.}04$ (more likely this will be on the order of $0\overset{''}{.}1$ at best), in addition to the

error caused by local deflections of the vertical. The correction, for use with longer arcs is:-

$$\text{(3-5)} \qquad d\,s = \frac{1}{8}\, a\, e^2\, (\Delta\varphi''\, \sin 1'')^3 \cos 2\,\varphi$$

Given astronomic latitudes for several stations in different arcs, as well as geodetically measured segment lengths s, we choose separate initial points for each arc. From the first two latitudes in any one arc we have:-

$$\text{(3-6)} \qquad \varphi_2 - \varphi_1 = \frac{S}{R \sin 1''} - \frac{d\,s}{R \sin 1''}$$

where

$$\text{(3-7)} \qquad \frac{1}{R} = \frac{(1 - e^2 \sin^2 \varphi)^{3/2}}{a\,(1 - e^2)}$$

In this method we make use of the used reference ellipsoid, for which we know the values a_o and e_o^2, and compute corrections δa and δe^2, to that ellipsoid resulting from the data given. Thus $a = a_o + \delta a$, and $e^2 = e_o^2 + \delta e^2$.

Expanding equation (3-7) into a series and substituting it in equation (3-5):-

$$\text{(3-8)} \qquad \varphi_2 - \varphi_1 = \frac{S}{\sin 1''}\left[\frac{1}{R_o} - \frac{\delta a}{a_o^2} + (1 - \frac{3}{2}\sin^2\varphi)\,\frac{\delta e^2}{a_o}\right] - \frac{ds}{R_o \sin 1''}$$

in which R_o is the value of R for a_o and e_o^2.

Considering the measured meridional arcs accurate in comparison with the observed latitudes, we give corrections v_1, v_2, to latitudes φ_1, φ_2, etc. Since in the derivation of (3-8), all terms containing e^2 were neglected in the coefficients of $\delta\, a$ and $\delta\, e^2$, i.e., e^2 was considered equal to zero, we can say $R = a$ and $s/(a \sin 1'') = \varphi_2 - \varphi_1$.

Thus equation (3-8) becomes:-

$$(\varphi_2+v_2) - (\varphi_1+v_1) = \frac{S}{R_o \sin 1''} + \frac{S}{a_o \sin 1''}\left[-\frac{\delta a}{a_o} + (1-\frac{3}{2}\sin^2\varphi)\delta e^2\right]$$

$$-\frac{d\ s}{R_o\ \sin 1''}\text{, and then}$$

$$\text{(3-9)}\quad v_2 = v_1 - \frac{\varphi_2 - \varphi_1}{a_o\ \sin 1''}\ \delta\ a + (1 - \frac{3}{2}\sin^2\varphi)\ (\varphi_2 - \varphi_1)\ \delta\ e^2$$

$$+\frac{s - ds}{R_o\ \sin 1''} - (\varphi_2 - \varphi_1).$$

This becomes the basic observation equation to be written for each segment of each arc. The unknowns are v_1, δ a, and $\delta\ e^2$.

In the least squares solution, in which $\Sigma\ v^2$ = minimum, the magnitudes of the various coefficient elements make it convenient to use the following modifications.

Placing $\quad x = \delta\ a/1000 \qquad y = 1000\ \delta\ e^2$

$$m_2 = -1000\ (\varphi_2 - \varphi_1)/a_o$$

$$n_2\ \frac{\varphi_2 - \varphi_1}{1000}\ (1 - \frac{3}{2}\sin^2\varphi)$$

$$l_2 = \frac{s - ds}{R_o\ \sin 1''} - (\varphi_2 - \varphi_1)$$

We can then write a simple error equation for each segment, substituting the above symbols in (3-9):-

$$v_2 = v_1 + m_2\ x + n_2\ y + l_2.$$

Since the initial latitude must be corrected, an additional observation equation is written for it, simply $v_1 = v_1$. In this method, there will be as many unknowns as there are arcs plus two, i.e., the v for the initial latitude of each arc, plus x and y. In the case

[illegible]

that a value for the initial v is assumed to be some arbitrary value, e.g. zero, there will be only two unknowns.

For each arc, observation equations will be as follow:-

$$
\begin{aligned}
&v_1 &&= v_1 \\
&v_1 + m_2x + n_2y + l_2 &&= v_2 \\
&v_1 + m_3x + n_3y + l_3 &&= v_3 \\
&\quad * * * * * \\
&v_1 + m_nx + n_ny + l_n &&= v_n
\end{aligned}
\tag{3-10}
$$

The result of a computation of this type presented by Crandall (6), and using data (circa 1890) given in the "Handbuch der Vermessungs-kunde" (37) for six arcs (French, English, Hanover, Prussian, Russian, and Swedish), varying in length from $0\overset{\circ}{.}5$ to $12\overset{\circ}{.}4$, is as follows:-

$$
\begin{aligned}
a &= 6\ 377\ 800 \text{ meters} \\
e^2 &= 0.0069091 \\
f &= 1/289
\end{aligned}
$$

The standard error of the mean amounts to $2\overset{''}{.}1$ for a latitude determination, which is quite a bit greater than the accuracy obtain-able for first order work, (about $0\overset{''}{.}1$). Thus, Crandall concludes (6), an ellipsoid of revolution will not fit the data without large local deviations of the vertical.

3.4 By Correction of Radius.

A method given by Bomford (2) p. 364, which is a variation of that described in Section 3.3, works with radii of curvature at points in an arc rather than with latitudes.

For given values of R in widely separated latitudes, the values of a and e^2 are obtained from the expansion of equation (2-2):-

$$(3\text{-}11) \qquad R = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 \varphi)^{3/2}}$$

$$= a\,(1 - e^2)\,(1 + \frac{3}{2} e^2 + \frac{3}{8} e^4 \sin^4 \varphi + \ldots)$$

Using $e^2 = 2f - f^2$ and the approximation $\delta e^2 \approx 2\,\delta f$, and dropping the terms of higher order than e^2, the following form is obtained:-

$$(3\text{-}12) \qquad \delta R = \delta a - 2a_o\, \delta f\, (1 - \frac{3}{2} \sin^2\varphi)$$

in which $\delta R = R_m - R_o$ results in changes δa and δf in a_o and f_o of the ellipsoid provisionally adopted. From equation (2-3) we may use $R_m = s \csc 1''/\Delta\varphi''$ for arcs shorter than about 100 miles (2). The value of R_o would be obtained for the used ellipsoid.

In a more practical application, more than two values of R would be available, and observation equations for least squares solution would be as follow:-

$$(3\text{-}13) \qquad v_1 = x - 2a_o y\,(1 - \frac{3}{2} \sin^2 \varphi_1) - \frac{s_1 \csc 1''}{\Delta\varphi_1''} + (R_o)_1$$

$$v_2 = x - 2a_o y\,(1 - \frac{3}{2} \sin^2 \varphi_2) - \frac{s_2 \csc 1''}{\Delta\varphi_2''} + (R_o)_2$$

* * * * *

$$v_n = x - 2a_o y\,(1 - \frac{3}{2} \sin^2 \varphi_n) - \frac{s_n \csc 1''}{\Delta\varphi_n''} + (R_o)_n$$

where $x = \delta a$ and $y = \delta f$ are the unknowns, s_1 is the measured length of the arc P_1P_2, and $\Delta\varphi_1'' = \varphi_2 - \varphi_1$ in seconds, etc. Again Σv^2 is made a minimum. Weights may be assigned as deemed necessary.

Chapter 4. Dimensions from Arcs, Using Deflections of the Vertical

4.1 Basic Considerations - Astro-Geodetic Deflections.

(Note: A more complete analysis of "basic considerations," involving absolute deflections of the vertical, is given in section 4.3.)

As mentioned earlier, a very convenient adaptation of the arc method makes use of astro-geodetic deflections of the vertical, which are generally available in profusion wherever triangulation control nets have been made. This has been done in many parts of the world in connection with the various national surveys being carried out.

This method can be applied to individual arcs of meridians in one extreme, and to the whole network of triangulation points in the area covered by a national control survey in the other. Because of the latter application of the method, Hayford chose to call it the "area method" when the results of the U.S.C. & G.S. computations of the figure of the earth based on data collected in the large area covered by the United States were published (10).

Here again, the reference ellipsoid used in computation of the geodetic data is used as a point of departure, or more accurately, an "area of departure", from which we must apply measurements and computations to get as close to the best ellipsoid as we can. This "best" ellipsoid is the one most nearly approximating the earth spheroid or mean geoid surface, in size, shape, and orientation.

Figure 2 shows in simple terms, the relationship which can exist between a reference ellipsoid and the geoid.

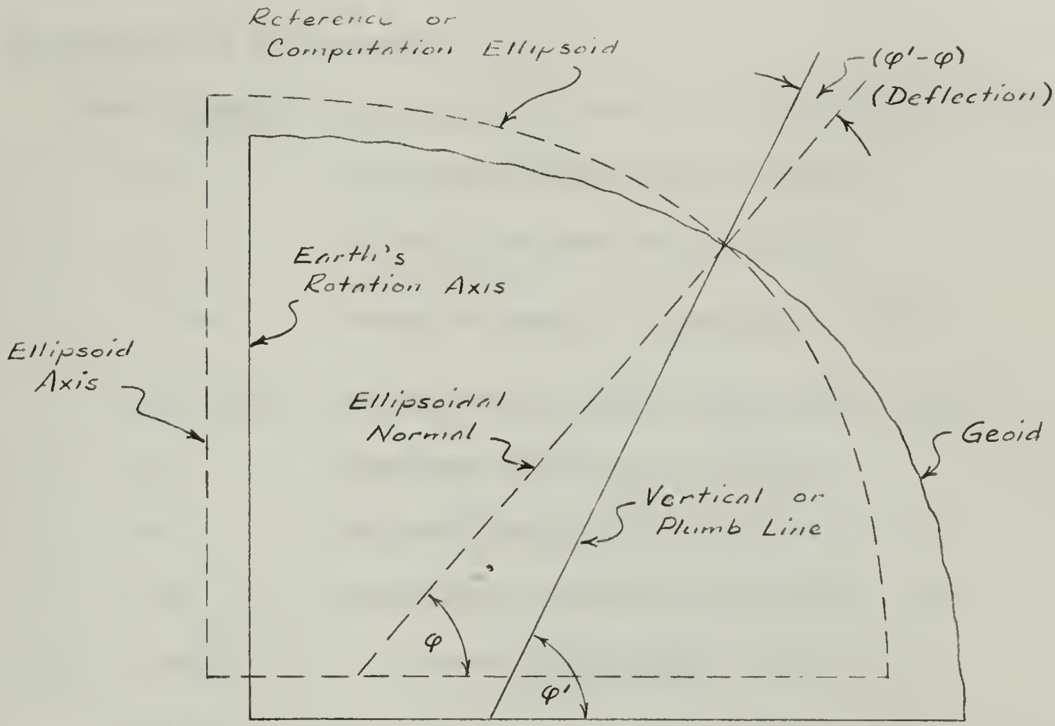

Fig. 2 - Astro-Geodetic Deflections of the Vertical

In general, there are six different items that may be corrected by the procedure involved, and the particular ones that are corrected depend upon what is desired and upon what and how much data is supplied. The existing geodetic data, i.e., latitudes, longitudes, deflections, etc.,depend upon the initial point (datum) of the survey, and upon the assumptions made about that point. The six items are: the equatorial radius a; the polar flattening f; and the following quantities assumed for the initial point: its latitude φ_o, its longitude λ_o, the geoid height N_o, and the azimuth A of its meridian. In many cases, all that are desired would be the corrections δa and δf. Usually the corrections to φ_o and λ_o at the initial point will be in the form of corrections to the meridian and prime vertical components to the

deflections of the vertical.

The following notation will be used in the discussion to follow:

φ', λ' - Astronomic latitude and longitude

φ, λ - Geodetic latitude and longitude

$(\varphi'-\varphi)$ - Meridian component, astro-geod. defl.

$(\lambda'-\lambda)\cos\varphi$ - Prime vertical comp., astro-geod. defl.

ξ - Residual deflection in meridian

η - Residual deflection in prime vertical

$\left.\begin{matrix} d\xi_D \\ d\eta_D \end{matrix}\right\}$ - Orientation corrections to deflections due to displacement of ellipsoid

$\left.\begin{matrix} d\xi_C \\ d\eta_C \end{matrix}\right\}$ - Change of ellipsoid corrections to deflections. These contain the values δa and δf which result when an ellipsoid is changed. These corrections actually apply to the geodetic parts of the astro-geodetic deflections.

When it is desired to find a better orientation of the ellipsoid as well as to find its size and shape, the observation equations which may be written for each point take the following general form:

$$\xi = (\varphi' - \varphi) + d\xi_D + d\xi_C \tag{4-1}$$

$$\eta = (\lambda' - \lambda) \cos \varphi + d\eta_D + d\eta_C$$

If it is desired only to find a and f, using data from a meridian arc only, the series of observation equations will be as follows:

(4-2) $$\xi_1 = (\varphi_1' - \varphi_1) + (d\,\xi_c)_1$$

$$\xi_2 = (\varphi_2' - \varphi_2) + (d\,\xi_c)_2$$

* * * * *

$$\xi_n = (\varphi_n' - \varphi_n) + (d\,\xi_c)_n$$

Then the least squares solution will involve making $\Sigma\xi^2 =$ minimum, this giving us the parameters a and f of an ellipsoid which will best fit the condition which, in effect, forces the residual deflections to a minimum, as close to zero as possible. Weights can be assigned to the observation equations as deemed necessary.

In general form, the condition can be stated:

(4-3) $$\sum(\xi^2 + \eta^2) = \text{minimum},$$

which is good for any problem. Either or both terms, ξ^2 and η^2, can be used according to the problem to be solved.

In this case, in which astro-geodetic deflections are used, the better results are obtained when these deflections have been reduced isostatically.

4.2 Formulas for Displacement and Change of Ellipsoid Corrections.

There are four well-known methods of computation of the change of ellipsoid and displacement corrections to the astro-geodetic deflections. These are based on formulas representing different derivation approaches to the problem. They are the formulas of Hayford (10), the U. S. Army Map Service (4), Bomford (2), and Vening Meinesz (33). The derivations of these formulas are given in the references cited (except (4)), and being beyond the scope of this paper, will not

be covered here.

4.21. Hayford's Formulas. The form of observation equation given in (10) is as follows:

$$(4\text{-}4)\quad \xi = k_1\,(\varphi) + l_1\,(\lambda) + m_1\,(A) + n_1\left(\frac{a}{100}\right) + o_1\,(10{,}000\ e^2) + (\varphi' - \varphi)$$

$$\eta = k_2\,(\varphi) + l_2\,(\lambda) + m_2\,(A) + n_2\left(\frac{a}{100}\right) + o_2\,(10{,}000\ e^2)$$

$$+ (\lambda' - \lambda)\cos\varphi$$

in which (φ), (λ), (A) are unknown corrections to the initial φ,λ, and A at the origin of the survey, and $\left(\frac{a}{100}\right)$ and $(10{,}000\ e^2)$ are unknown corrections to the values of a and e^2 for the reference ellipsoid. It can be seen that these are similar in form to equations (4-1).

The change of ellipsoid corrections are:

$$(4\text{-}5)\quad d\xi_c = n_1\left(\frac{a}{100}\right) + o_1\,(10{,}000\ e^2)$$

$$d\eta_c = n_2\left(\frac{a}{100}\right) + o_2\,(10{,}000\ e^2)$$

in which $n_1 = \dfrac{100}{a \sin 1''}\,\dfrac{N}{R}\,\theta\,\cos A_B$

$$\theta = \frac{S}{N}\left[1 + \frac{e^2\,\theta^2}{6(1-e^2)}\cos^2\varphi_o\,\cos^2 A_F\right]$$

$$o_1 = \frac{N}{R}\;\frac{\sin^2\varphi_o}{20{,}000 \sin 1''\,(1-e^2\sin^2\varphi_o)}\,\theta\,\cos A_B$$

$$-\frac{R}{N}\;\frac{1}{40{,}000 \sin 1''\,(1-e^2)^2}\,(\varphi-\varphi_o)[1+2\sin^2\varphi_o + 3\cos(\varphi_o+\varphi)]$$

$$n_2 = -\frac{100}{a \sin 1''}\,\theta\,\sin A_B$$

$$o_2 = -\frac{\sin^2\varphi_o}{20{,}000 \sin 1''(1-e^2\sin^2\varphi_o)}\,\theta\,\sin A_B$$

$$+\frac{(1-e^2\sin^2\varphi_o)^{1/2}\cos^2\varphi_o}{60{,}000\ a \sin 1''\,(1-e^2)^2}\;S\,\theta^2\cos^2 A_F\,\sin A_B$$

The corrections for displacement (orientation) are:-

$$d\xi_D = k_1\ (\varphi) + l_1\ (\lambda) + m_1\ (A) \tag{4-6}$$

$$d\eta_D = k_2\ (\varphi) + l_2\ (\lambda) + m_2\ (A)$$

in which

$$k_1 = -1 + \frac{S}{R}(1 + Q)\ \frac{\sin^2(\lambda - \lambda_o)}{2 \sin^2 \frac{1}{2}(A_B - A_F)}$$

$$l_1 = \text{zero}$$

$$m_1 = \frac{S}{R}(1 + Q)\ \frac{\sin A_B\ [1 + \cos(\lambda - \lambda_o)]}{2 \sin^2 \frac{1}{2}(A_B - A_F)}$$

$$k_2 = -\sin\varphi \sin(\lambda - \lambda_o)$$

$$l_2 = -\cos\varphi$$

$$m_2 = \frac{\cos\varphi \cos A_B \sin(\lambda - \lambda_o)}{\sin A_F}$$

$$Q = \frac{\theta^2}{12} \cos^2 \frac{1}{2}(A_B - A_F)$$

The purpose of the coefficients k, l, m, n, and o can be stated as in this example for one of them: k_1 is a numerical coefficient, such that if the latitude of the initial point were corrected by the amount (φ), the change produced in $\varphi' - \varphi$ would be $k_1\ (\varphi)$.

It can be seen from inspection of the coefficients that for points on the same meridian as the initial point, the values are $k_1 = -1$, $l_1 = 0$, and $m_1 = 0$, so that the observation equation would be:-

$$\xi = (\varphi) + n_1\left(\frac{a}{100}\right) + o_1\ (10{,}000\ e^2) + (\varphi' - \varphi). \tag{4-7}$$

4.22 Army Map Service Formulas. In an article in Transactions of the American Geophysical Union, No. 37, 1956, Chovitz and Fischer

[illegible]

$$[illegible]$$

[illegible]

[illegible]

reported on the U. S. Army Map Service's determination of the figure of the earth from arcs (4). The procedure is described in some detail, the essential part regarding correction formulas being reproduced in somewhat different form here. The method is designed to be correct for meridional arc segments of 1^{o}, but not more than 3^{o} in length, and for arcs of parallel of comparable length.

For meridian arcs the formulas are:

$$(4\text{-}7) \qquad d\xi_C = \left[\frac{\delta a}{a}(\varphi_n - \varphi_o) + \delta e^2 \left\{ I(\varphi_n - \varphi_o) - \sum_{n=1}^{n} (\varphi_n - \varphi_{n-1}) \left[II \cos 2\varphi_m - III \cos 4\varphi_m + IV \cos 6\varphi_m \right] \right\} \right] \csc 1''$$

in which φ_m = mid-latitude of segment $= \frac{1}{2}(\varphi_{n-1} + \varphi_n)$

$$(4\text{-}8) \qquad d\xi_D = \xi_o - (\varphi_o' - \varphi_o) = \text{initial orientation of the arc segment.}$$

In this method the computation proceeds segment by segment, with a new initial point for each segment of the arc when the length limits given above are reached. Thus the procedure is similar to the one described in section 3.3. This presupposes that all the points in the arc furnish data that is adjusted to the same datum. In contrast, the Hayford formulas (and those to be given subsequently) are designed to cover a large, but limited area, using only one initial point in the computation. The unknowns in formulas (4-7) and (4-8) are ξ_o, δa, and δe^2. The Roman numeral coefficients are:

$$I = -\frac{1}{4} - \frac{7}{16}e^2 - \frac{17}{32}e^4 \qquad IV = \frac{3}{64}e^4$$

$$II = \frac{3}{4} + \frac{3}{4}e^2 + \frac{45}{64}e^4$$

$$III = \frac{3}{16}e^2 + \frac{9}{32}e^4$$

For arcs of parallel the formulas are:

$$(4\text{-}9)\quad d\eta_C = \Big[\frac{\delta a}{a}(\lambda_n-\lambda_o)\cos\varphi + \delta e^2(\lambda_n-\lambda_o)\,\frac{\sin^2\varphi\cos\varphi}{2}\,(1+e^2\sin^2\varphi + e^4\sin^4\varphi)\Big]\csc 1''$$

where φ = latitude of the parallel.

$$(4\text{-}10)\quad d\eta_D = \eta_o-(\lambda'_o - \lambda_o)\cos\varphi = \text{initial orientation of the arc segment}$$

The unknowns are η_o, δa, and δe^2. Solutions for meridians and parallels can be done separately or can be combined for a cross-solution (4).

4.23. Bomford's Formulas. The formulas for change of ellipsoid given in Bomford (2) p. 127-130, actually contain corrections for both displacement and change of ellipsoid. They are developed here in separated form. These formulas are used in Bomford's version of the area method, which supposes a "homogeneously computed national survey" covering a 20° by 20° area containing 50 astro-geodetic stations. Bomford states that it will generally be best to ignore values of η deduced from azimuth observations (these were omitted in the description of Hayford's method above), and to use only those based on longitude (2) p. 365.

The change of ellipsoid formulas as given by Bomford (2) p. 129, 365, are as follows:

$$(4\text{-}11)\quad d\xi = -\Big[(1/a)\sin u\cos\omega\,(U\sin u_o + V\cos u_o) + \sin u\sin\omega\,\delta\eta_o - (1/a)\cos u(V\sin u_o - U\cos u_o) + \sin 2u\;\delta f\Big]\csc 1''$$

(4-12) $$d\eta = -\left[(1/a)\sin\omega\,(U\sin u_o + V\cos u_o) - \cos\omega\,\delta\eta_o\right]\csc 1''$$

where $U = -a\,\delta\xi_o - a\,\delta f\sin 2u_o$

$V = N_o - \delta a + a\,\delta f\sin^2 u_o$

u = reduced latitude; $\tan u = (1-f)\tan\varphi$

ω = longitude measured east from origin

N_o = height of origin in corrected ellipsoid above that in reference ellipsoid

$\delta\xi_o$ and $\delta\eta_o$ are in radians.

Substituting the values of U and V in (4-11) and (4-12), and separating the expressions into two parts: $d\xi = d\xi_C + d\xi_D$ and $d\eta = d\eta_C + d\eta_D$, we obtain:

(4-13) $$d\xi_C = \Big[(\sin u\cos\omega\cos u_o - \cos u\sin u_o)\ \delta a/a + \big[\sin u\cos\omega\sin^2 u_o\cos u_o + \cos u\sin u_o(1+\cos^2 u_o) - \sin 2u\big]\delta f\Big]\csc 1''$$

(4-14) $$d\xi_D = (\sin u\cos\omega\sin u_o + \cos u\cos u_o)\,\delta\xi_o'' - \sin u\sin\omega\,\delta\eta_o'' - (\sin u\cos\omega\cos u_o - \cos u\sin u_o)(N_o/a)\csc 1''$$

(4-15) $$d\eta_C = \left[\sin\omega\cos u_o(\delta a/a) + \sin\omega\sin^2 u_o\cos u_o\ \delta f\right]\csc 1''$$

(4-16) $$d\eta_D = \sin\omega\sin u_o\,\delta\xi_o'' + \cos\omega\,\delta\eta_o'' - \sin\omega\cos u_o(N_o/a)\csc 1''$$

4.24. Vening Meinesz' Formulas. The formulas here referred to are those published in 1950 (33), and which were developed for the

purposes of (a) describing the relationship between the variation of the vertical deflection components at the origin of a survey and the changes resulting in the deflections at any arbitrary point in the survey, the variation having been caused by a parallel displacement of the reference ellipsoid with respect to the geoid; and (b) describing the changes in the deflections of the vertical brought about by a change of reference ellipsoid.

The complete formulas referred to in (a) above, are as follow:-

$$(4\text{-}17)\quad d\xi_D = \frac{R_o}{R}\left[\cos(\varphi-\varphi_o) - 2\sin\varphi\sin\varphi_o\sin^2\tfrac{1}{2}(\lambda-\lambda_o)\right]\delta\xi_o + \frac{r_o}{R}\sin\varphi\sin(\lambda-\lambda_o)\,\delta\eta_o + \frac{1}{R}\left[\sin(\varphi-\varphi_o) - 2\sin\varphi\cos\varphi_o\sin^2\tfrac{1}{2}(\lambda-\lambda_o)\right]\delta N_o$$

$$(4\text{-}18)\quad d\eta_D = -\frac{R_o}{r}\sin\varphi_o\sin(\lambda-\lambda_o)\,\delta\xi_o + \frac{r_o}{r}\cos(\lambda-\lambda_o)\,\delta\eta_o - \frac{1}{r}\cos\varphi_o\sin(\lambda-\lambda_o)\,\delta N_o$$

where R_o, R = radius of curvature in meridian at initial point P_o and at point P, respectively

r_o, r = radius of curvature in prime vertical at initial point P_o and at point P, respectively (r is used here, rather than the usual N, which is reserved for geoid distance).

The above form is that given by Heiskanen in (13) p. 302, and is better suited to the use of tables than the form in the original paper (33), which uses $a(1-e^2)/W^3$ rather than R, and a/W rather than

r, where $W = (1-e^2 \sin^2\varphi)^{\frac{1}{2}}$.

Vening Meinesz points out that his formulas are similar to those developed by Helmert (37), but include δN_o terms as essential differences.

If the line P_oP is long, the complete formulas above must be used. However, for nets of small east-west extent, terms proportional to the squares of $(\lambda-\lambda_o)$ can be neglected; and for nets of small extent in latitude, the ratios R_o/R, r_o/R, R_o/r, and r_o/r are very nearly 1 (within 0.01), and the formulas (4-17) and (4-18) become:-

$$(4\text{-}19)\quad d\xi_D = \cos(\varphi-\varphi_o)\delta\xi_o + \sin\varphi \sin(\lambda-\lambda_o)\delta\eta_o + \sin(\varphi-\varphi_o)\frac{\delta N_o}{a}$$

$$(4\text{-}20)\quad d\eta_D = -\sin\varphi_o \sin(\lambda-\lambda_o)\,\delta\xi_o + \delta\eta_o - \cos\varphi_o \sin(\lambda-\lambda_o)\frac{\delta N_o}{a}$$

According to (13), the formulas (4-19) and (4-20) apply when $P_oP <$ 300 km. in east-west direction and < 1500 km. in north-south direction. However, Vening Meinesz (33) states that if it is desired not to neglect quantities greater than 1/300 of the deflection of the vertical, these formulas can be used for nets extending up to 4° 40' in east-west direction and to more than 20° in latitude, from the origin P_o. Furthermore, the complete formulas will seldom require that the ratios between R, R_o, r, and r_o be anything but 1 even for large nets, to attain the required accuracy.

The complete formulas referred to in (b) above, for change of ellipsoid, are here transcribed:-

$$(4\text{-}21)\quad d\xi_c = \Big\{[\sin(\varphi-\varphi_o) - 2\cos\varphi_o \sin\varphi \sin^2\tfrac{1}{2}(\lambda-\lambda_o)]\,\delta\beta$$

$$- 4\cos\varphi\cos\tfrac{1}{2}(\varphi+\varphi_o)\sin\tfrac{1}{2}(\varphi-\varphi_o)\,\delta f$$

$$+ [(2-3\sin^2\varphi - \cos(\varphi+\varphi_o))\sin(\varphi-\varphi_o)$$

$$- 2\cos\varphi_o\sin\varphi\,(2+\sin^2\varphi_o - 3\sin^2\varphi)\sin^2\tfrac{1}{2}(\lambda-\lambda_o)$$

$$- \sin 2\varphi_o \sin^2\tfrac{1}{2}(\varphi-\varphi_o) - 2\cos\varphi\cos\tfrac{1}{2}(\varphi+\varphi_o)$$

$$\sin\tfrac{1}{2}(\varphi-\varphi_o)]\,f\delta\beta + [\tfrac{1}{2}\sin 4\varphi_o\,(\sin^2\tfrac{1}{2}(\varphi-\varphi_o)$$

$$+ \sin\varphi_o\sin\varphi\sin^2\tfrac{1}{2}(\lambda-\lambda_o)) - \sin\varphi_o\sin\tfrac{3}{2}(\varphi+\varphi_o)$$

$$\sin\tfrac{3}{2}(\varphi-\varphi_o) - \sin\varphi_o\sin\tfrac{1}{2}(\varphi+\varphi_o)\sin\tfrac{1}{2}(\varphi-\varphi_o)$$

$$- 4\cos\varphi\cos\tfrac{1}{2}(\varphi+\varphi_o)\sin\tfrac{1}{2}(\varphi-\varphi_o)$$

$$+ 4\sin 2\varphi\cos^2\tfrac{1}{2}(\varphi+\varphi_o)\sin^2\tfrac{1}{2}(\varphi-\varphi_o)]f\delta f\Big\}\ \mathrm{csc}1''$$

$$(4\text{-}22)\quad d\eta_c = \Big\{-\cos\varphi_o\sin(\lambda-\lambda_o)\,\delta\beta$$

$$+ \cos\varphi_o\sin(\lambda-\lambda_o)\,[\sin(\varphi-\varphi_o)\sin(\varphi+\varphi_o)\,f\delta\beta$$

$$+ \tfrac{1}{4}\tan\varphi_o\sin 4\varphi_o\,f\delta f]\Big\}\ \mathrm{csc}1''$$

where $\delta\beta = \dfrac{\delta a}{a} + \sin^2\varphi_o\,\delta f$

The formula (4-21) is quite complicated, but can be simplified for systems not of world-wide extent. When the terms $f\delta f$ and $f\delta\beta$ are assumed to have a magnitude of 1:6,000,000 or less, the terms containing the squares and products of $(\varphi-\varphi_o)$ and $(\lambda-\lambda_o)$ may be

neglected in the factors by which the terms are multiplied. When these angles are such that the distance from the origin is 600 km., the squares and products amount to an order of magnitude of 1:600,000,000 (33).

The simplified formulas are:-

$$(4\text{-}23)\quad d\xi_c = \Big\{[\sin(\varphi-\varphi_o) - 2\cos\varphi_o \sin\varphi \sin^2 \tfrac{1}{2}(\lambda-\lambda_o)]\delta\beta$$
$$- 4\cos\varphi \cos\tfrac{1}{2}(\varphi+\varphi_o)\sin\tfrac{1}{2}(\varphi-\varphi_o)\delta f$$
$$- (2 + \tfrac{3}{4}\tan\varphi_o \sin 4\varphi_o)\sin(\varphi-\varphi_o) f\delta f\Big\} \csc 1''$$

$$(4\text{-}24)\quad d\eta_c = [-\cos\varphi_o \sin(\lambda-\lambda_o)\delta\beta$$
$$+ \tfrac{1}{4}\sin\varphi_o \sin 4\varphi_o \sin(\lambda-\lambda_o) f\delta f] \csc 1''$$

4.3 The Use of Gravimetric Deflections of the Vertical.

Probably the most accurate method of determining the equatorial radius from deflections of the vertical, involves a final correction by application of the gravimetrically determined deflections of the vertical ("absolute" deflections).

The following analysis could have been given at the beginning of this chapter, but it was felt better to place it here, thus keeping the complete picture intact.

Rice (25) p. 2, states that "a fundamental geodetic problem is to arrive at a reference ellipsoid of such dimensions and having such orientation that the deflections of the vertical are as small as

possible in the aggregate."

An ideal reference ellipsoid would be centered at the center of mass of the earth and its polar axis would coincide with the mean axis of rotation of the earth. This is what Veis (32) p. 16, calls the Terrestrial Ellipsoid. A further requirement which would be placed on an ideal reference ellipsoid would be that its surface best fit the surface of the geoid.

The origin or initial point of a geodetic system is generally assigned the astronomic latitude and longitude, φ'_o and λ'_o, observed at the place. The geoid distance N'_o is generally considered zero and no errors in azimuth are assumed. The effect of making $\varphi = \varphi'$, and $\lambda = \lambda'$ at the initial point is a parallel displacement of the reference ellipsoid's system of coordinates with respect to those of the terrestrial ellipsoid and the geoid. At another point, P_1, in the same geodetic system, which is displaced from the geoid, the normals do not coincide, the result not only of the displacement, but of the wrong computation surface, which is not the same size or shape as the true earth spheroid. The resulting meridian angle between φ' and φ is the astro-geodetic deflection $(\varphi' - \varphi)$ in the direction of the meridian. (This discussion is generally in terms of this meridian component, for convenience, though similar remarks can be made concerning the prime vertical component, as well as the total deflection).

The meridian component and the prime vertical component of the astro-geodetic deflection each consists of three angles in the same

plane, with respect to the normal to the mean or smoothed geoid. If we can correct this deflection in such a manner that both components are reduced to zero, what would remain would be the normal to the mean geoid. In doing so we have to change the size, shape, and position of the reference or computation ellipsoid so that our point P_1 coincides (as nearly as possible) with the corresponding point on the mean geoid, and so the new ellipsoid is in the position of a terrestrial ellipsoid. The three angles are:-

(1) Correction for displacement, $(d\xi_D, d\eta_D)$

(2) Correction for change of ellipsoid, $(d\xi_c, d\eta_c)$

(3) The absolute deflection of the vertical, or the angle between the true vertical and the normal to the mean geoid, or if preferred, the terrestrial ellipsoid of best fit. Since this is obtained gravimetrically, we call its two components ξ_g and η_g.

See figures 3 and 4 for pictorial descriptions of this.

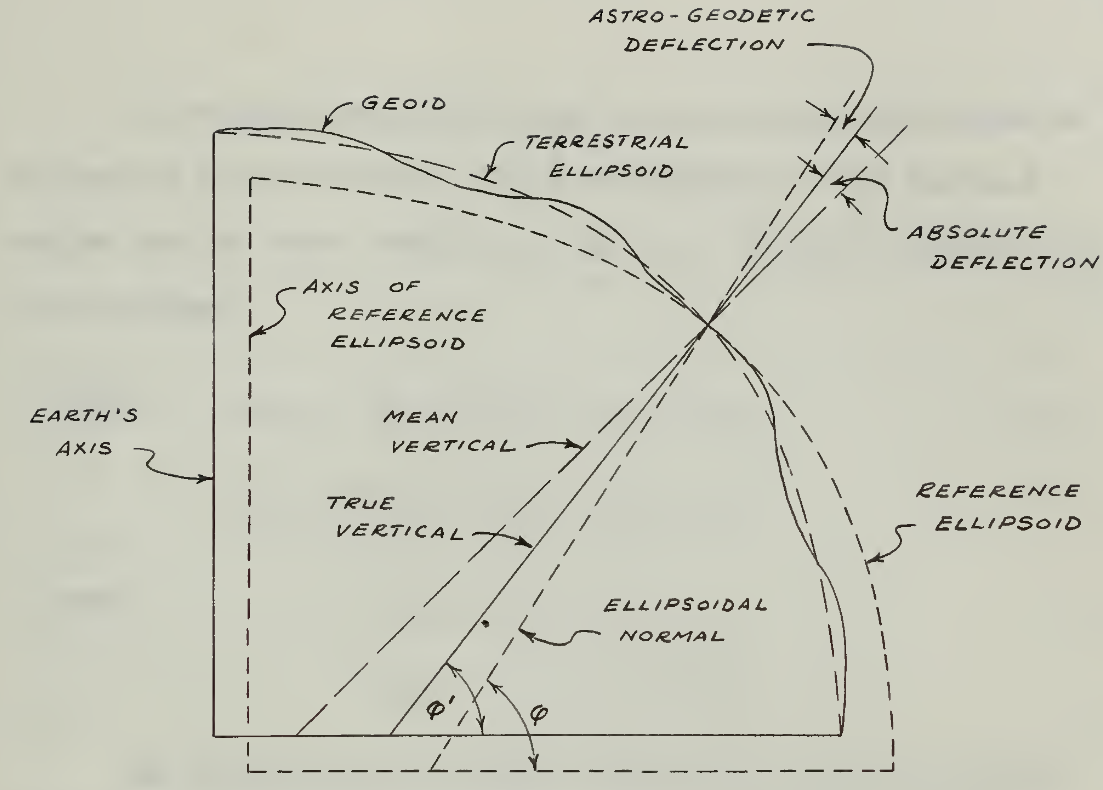
ASTRO-GEODETIC DEFLECTION
GEOID
TERRESTRIAL ELLIPSOID
ABSOLUTE DEFLECTION
AXIS OF REFERENCE ELLIPSOID
EARTH'S AXIS
MEAN VERTICAL
TRUE VERTICAL
REFERENCE ELLIPSOID
ELLIPSOIDAL NORMAL
φ'
φ

Fig. 3

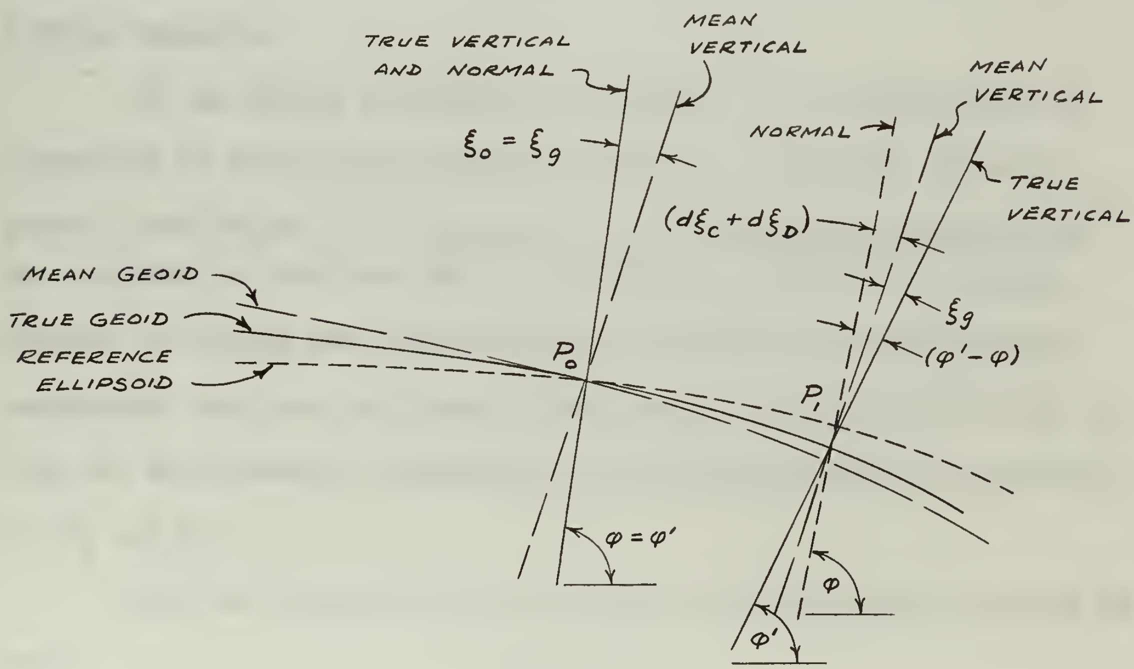
TRUE VERTICAL AND NORMAL
MEAN VERTICAL
$\xi_0 = \xi_g$
NORMAL
MEAN VERTICAL
TRUE VERTICAL
$(d\xi_c + d\xi_D)$
ξ_g
$(\varphi' - \varphi)$
MEAN GEOID
TRUE GEOID
REFERENCE ELLIPSOID
P_0
P_1
$\varphi = \varphi'$
φ
φ'

Fig. 4

It should be noted that when the gravimetric deflections at the initial point are known, two of the unknowns of the previous problem are now known, namely $\delta\xi_o$, and $\delta\eta_o$. These are equivalent to the following:-

(4-25) $$\delta\, \xi_o = -\,(\xi_a - \xi_g)_o = (\xi_g - \xi_a)_o$$

$$\delta\, \eta_o = -\,(\eta_a - \eta_g)_o = (\eta_g - \eta_a)_o$$

where $$(\xi_a)_o = (\varphi_o' - \varphi_o)$$

$$(\eta_a)_o = (\lambda_o' - \lambda_o)\cos\varphi_o$$

The gravimetrically computed quantities ξ_g and η_g, are independent of the size of the reference ellipsoid (13) p. 239, even though they depend on gravity anomalies derived from the gravity formula employed.

In the method (described in section 4.1) of determining the dimensions of the earth ellipsoid by means of correcting the astro-geodetic deflections with $d\xi_D$ and $d\xi_c$, and subsequently reducing $\Sigma\xi^2$ to a minimum, we have one effective method of solving the problem. However, we can go one step farther, as indicated in the preceding paragraphs, and make the final correction by subtracting the value ξ_g from the astro-geodetic deflections which have already been corrected by $d\xi_D$ and $d\xi_c$.

Thus our problem will be solved by a least squares solution in which

$$\Sigma\ (\xi - \xi_g)^2 = \text{minimum}$$
$$\Sigma\ (\eta - \eta_g)^2 = \text{minimum}$$

In this way we are much closer to the mean geoid at the start. The result should be a terrestrial ellipsoid which best fits the surface of the geoid, providing, of course, that we have sufficient points from a large enough area for our computation. Theoretically the problem can be solved when astro-geodetic and gravimetric deflections are known at just two points, one at each end of a long arc, preferably as long as the earth's radius. Naturally, the more points at which this data is known the better the solution. Heiskanen has suggested this method in a number of publications; see particularly (12), and (13) p. 278-279.

Chapter 5. Flattening from Gravity Anomalies.

A value for flattening can be gotten from the solution by the astro-gravimetric method described in section 4.3, but if this is done the gravity anomalies used in computation of the gravimetric deflections are no longer correct since these anomalies are based on the adopted gravity formula, which in turn is based on a particular ellipsoid with its own flattening. This gives a clue as to how the flattening may be corrected from the anomalies themselves. See (13), p. 279.

According to the theorem of Clairaut (13) p. 52, the following is true (extended form):-

$$\text{(5-1)} \qquad \beta = \frac{\gamma_p - \gamma_E}{\gamma_E} = \frac{5}{2} m - f - \frac{17}{14} mf$$

where γ_p, γ_E are normal gravity at pole, equator, resp.

m = (centrifugal force at equator)/$\gamma_E = \dfrac{\omega^2 a}{\gamma_E}$

f = flattening

The basic form of the gravity formula giving normal gravity γ for any latitude φ on the ellipsoid is (neglecting the longitude term):

$$\text{(5-2)} \qquad \gamma = \gamma_E \left(1 + \beta \sin^2\varphi - \epsilon \sin^2 2\varphi\right)$$

where $\epsilon = \frac{5}{8} mf - \frac{1}{8} f^2$

The problem is to correct the parameters γ_E, β, and ϵ, to the gravity formula where these parameters have been computed for a particular reference ellipsoid. Thus if β and ϵ are corrected, a

correction to f is automatically found. The correction to ϵ (based on currently used ellipsoids) is always so small, we may neglect its computation.

Using, for example, the International Ellipsoid, the following values are used as a basis:-

$a = 6\ 378\ 388$ m. $\qquad f = 1/297 = 0.0033670$

$\gamma_E = 978.049$ gal $\qquad m = 1/288.36 = 0.0034678$

$\beta = 0.0052884$ $\qquad \epsilon = 0.0000059$

For obtaining a corrected value of γ, we have:-

$$\gamma' = \gamma + \Delta\gamma$$

where $\quad \Delta\gamma = x' + y' \sin^2 \varphi$

in which x' = unknown correction to γ_E, in mgals.

y' = unknown correction to β, in mgals.

Before correction, the gravity anomalies, based on the ellipsoid used and isostatically reduced, are:-

$$\Delta g = g_o - \gamma$$

After correction they are:-

$$v = g_o - \gamma' = g_o - (\gamma + \Delta\gamma)$$

$$(g_o - \gamma) - \Delta\gamma = \Delta g - \Delta\gamma$$

Then, we can write error equations for each point where gravity observations have been made:-

$$x' + y'\sin^2\varphi - \Delta g = -v$$

If we have enough such stations, spread out over a large

enough area, we can solve the problem effectively by the method of least squares in such a manner that:-

$$\Sigma v^2 = \text{minimum}.$$

An example of this type of solution made by Heiskanen, involved using anomalies from over 1500 squares of 1° by 1° dimensions (13) p. 77. The resulting parameters were:-

$$\gamma_E = 978.0451$$

$$\beta = 0.0053026$$

$$\epsilon = 0.0000059$$

The flattening corresponding to the corrected β was:-

$$f = 1/298.2.$$

Chapter 6. Gravimetric Deflections of the Vertical

6.1 General.

The gravimetric method, of which the subject of Chapter 5 is a part, also produces what are sometimes referred to as "absolute" deflections of the vertical, obtained from gravity anomalies. The absolute or gravimetric deflection of the vertical at a point is defined as the angle between the normal to the geoid and the normal to the earth spheroid at the point. -- Veis (32) uses the term "mean geoid" to describe the surface of the earth spheroid in the area of a point, meaning a smooth surface striking a "mean" between undulations of the geoid. -- The discussion in Chapter 4 (section 4.3), describes the use of these deflections in determining the size of the earth ellipsoid. This chapter will present a brief outline of the method of determining the deflections from gravity anomalies.

To begin with, the gravity anomalies Δg, are caused by disturbing mass anomalies, and with this in mind and with the aid of fig. 5, we can easily see how local absolute deflections occur.

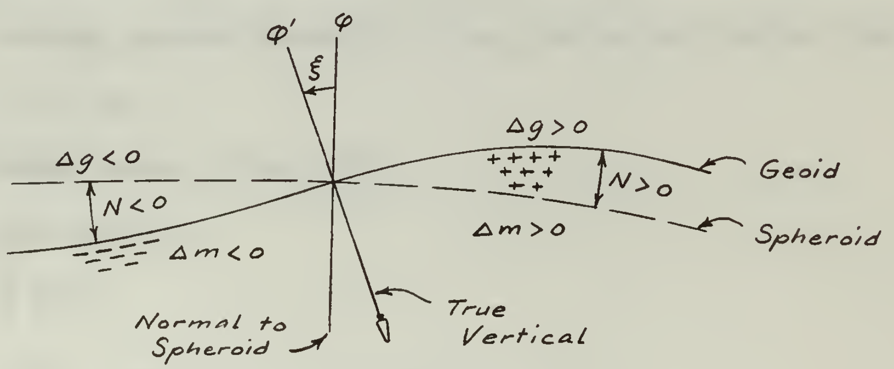

Fig. 5

The mass surplus tends to "pull" the plumb line toward it, and conversely the mass deficiency tends to repel it, thus causing the deflection $\xi = \varphi' - \varphi$. The sign of this deflection is apparent from this expression which shows that ξ is always referred to the astronomic latitude, φ', of the point in question. The sign convention for gravimetric deflections is the same as for astro-geodetic deflections. Referring to figure 5, we wee that when $\varphi > \varphi'$, $\xi = (\varphi' - \varphi) < 0$, and when $\varphi < \varphi'$, $\xi = (\varphi' - \varphi) > 0$. In U.S. practice λ is considered positive westward, and in European practice positive eastward. In either case, when $\lambda > \lambda'$, $\eta = (\lambda' - \lambda) \cos \varphi < 0$, and when $\lambda < \lambda'$, $\eta = (\lambda' - \lambda) \cos \varphi > 0$.

From a theorem developed by Stokes, Vening Meinesz formulated expressions for absolute deflections of the vertical in 1928. Stokes' Formula expressed the geoid distances N, as follows:-

$$(6\text{-}1) \qquad N = \frac{R}{2\,\pi\,G} \int_0^{2\pi} d\,A \int_0^{\pi} f(\psi) \sin \psi\, \Delta g\, d\,\psi$$

in which ψ = angular distance of a circular ring of width $d\psi$. See fig. 6.

Δg = average gravity anomaly in ring, having the limits ψ and $\psi + d\psi$.

R = Mean radius of the earth.

G = Mean gravity.

$f(\psi) = \frac{1}{2} S(\psi)$.

$$(6\text{-}2) \qquad S(\psi) = \csc \frac{1}{2}\psi + 1 - 6 \sin \frac{1}{2}\psi - 5 \cos \psi -$$

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[illegible]

$$- 3 \cos \psi \; l \; n \; (\sin \frac{1}{2} \psi + \sin^2 \frac{1}{2} \psi)$$

which is the original Stokes Function.

$\sin \psi \, d\psi \, dA$ = surface area element for unit sphere.

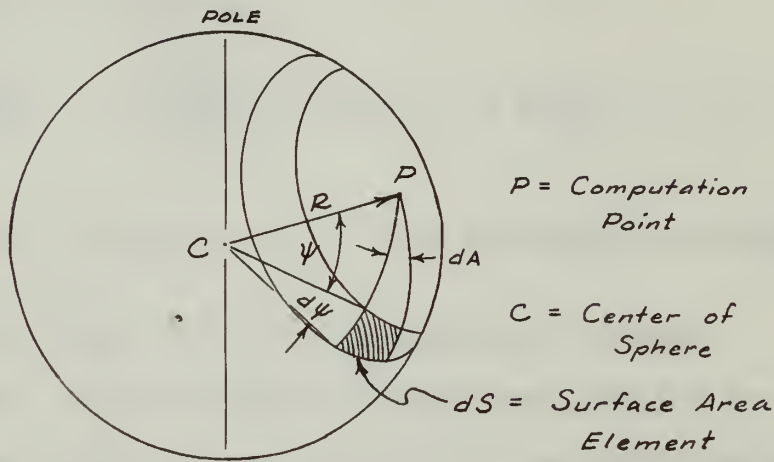

Fig. 6

Vening Meinesz got his formulas by differentiating the Stokes formula, as indicated in fig. 7, in the form:-

$$(6\text{-}3) \qquad N = \frac{1}{2 \pi G R} \int_S f(\psi) \, \Delta g \, d \, S$$

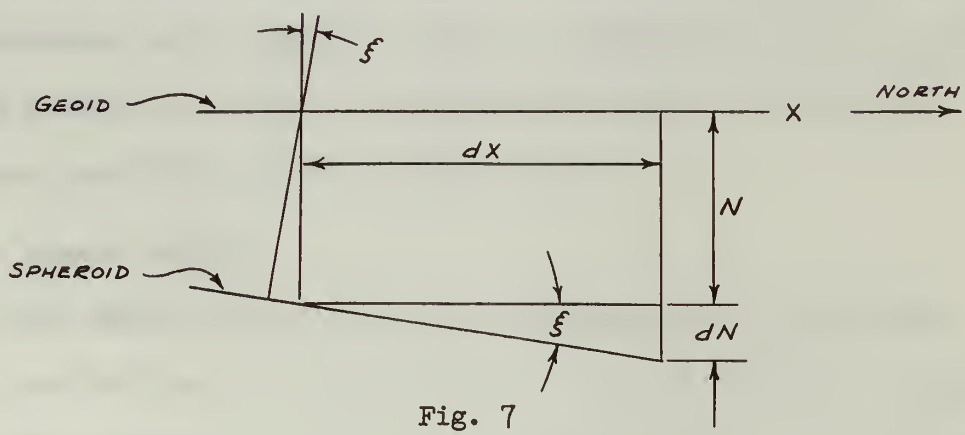

Fig. 7

Upon differentiating, the following are obtained:-

$$(6\text{-}4)\quad d\xi'' = -\frac{\delta N}{\delta x} = \frac{\csc 1''}{2\pi G R^2}\int_s f'(\psi)\cos A\,\Delta g\,dS$$

$$(6\text{-}5)\quad d\eta'' = -\frac{\delta N}{\delta y} = \frac{\csc 1''}{2\pi G R^2}\int_s f'(\psi)\sin A\,\Delta g\,dS$$

where $\quad dS = R^2 \sin\psi\, d\psi\, dA$, and

$$(6\text{-}6)\, f'(\psi) = \frac{df(\psi)}{d\psi} = -\frac{\cos\frac{1}{2}\psi}{2\sin^2\frac{1}{2}\psi} + 8\sin\psi - 6\cos\tfrac{1}{2}\psi$$

$$-\,3(1-\sin\tfrac{1}{2}\psi)\csc\psi + 3\sin\psi\ln\left[\sin\tfrac{1}{2}\psi\,(1+\sin\tfrac{1}{2}\psi)\right]$$

Application of these formulas to practical use requires further manipulation. They are used in different form, whether one is going to compute the effect of areas beyond $\psi \approx 3^\circ$, or within the area circumscribed by a circle of radius $\psi \approx 3^\circ$ (3° is an arbitrary dividing line, which, though close to the limit, has been chosen for convenience (31) p. 96). These different forms are described as the "Circle-Ring" Method, for use up to $\psi \approx 3^\circ$, and the "Squares" Method, for use beyond. Still another form is used for the small inner circle close to the computation point, having a radius r_o, generally, of 5 km. or less. Figure 8 shows this division of methods pictorially. For further references, see (13) p. 63-70, 257-264, (15).

6.2 The Squares Method.

This method, used for the earth represented as a spherical surface, and for the area beyond a radius of about 3° from the computation point, makes use of mean gravity anomaly values for each 1° by 1° square (spherical trapezoid) between $\psi \approx 3^\circ$ and $\psi \approx 20^\circ$, and for each 5° by 5° square beyond 20° and around to the antipode (31).

[illegible]

$$[illegible]$$

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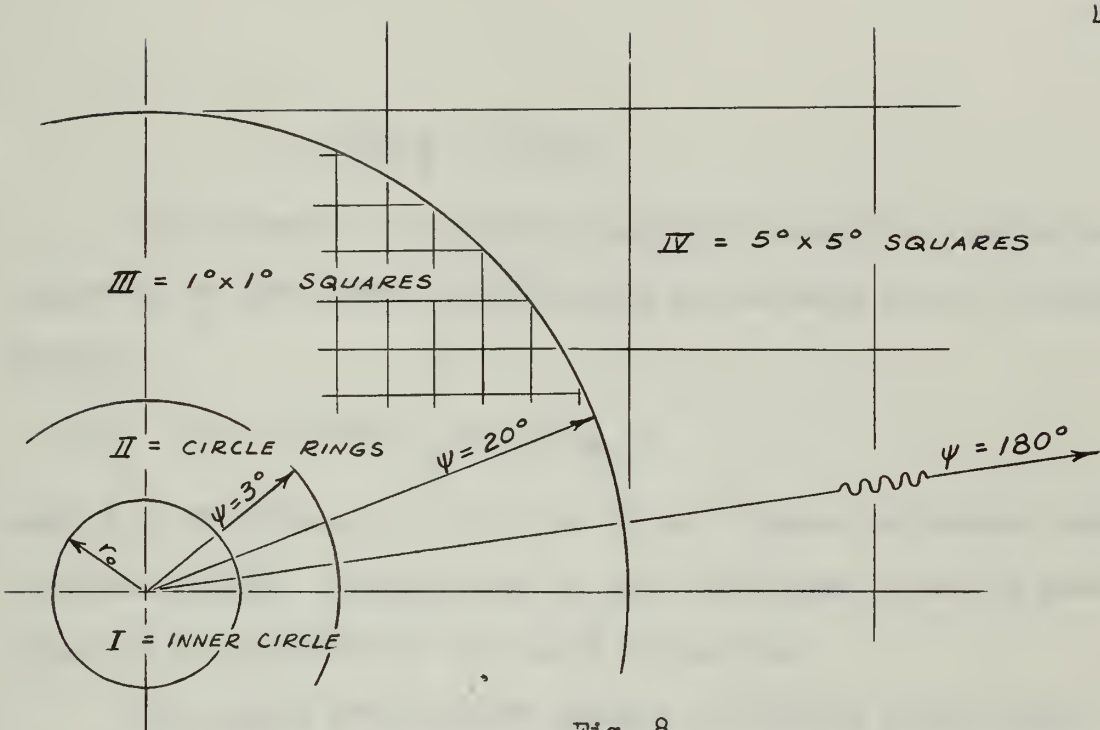


Fig. 8

These should be either free-air or isostatic anomalies, the latter being best. If Bouguer anomalies are used they must be corrected to obtain free-air anomalies. To adapt formula (6-4) to the squares method, we first replace dS by $R^2 \sin \psi \; d \psi \; d A$, obtaining:-

$$(6\text{-}7) \qquad d\xi'' = \frac{\csc 1''}{2 \pi G} \int_s f'(\psi) \cos A \, \Delta g \sin \psi \, d \psi \, d A$$

Then using the square element dq on the unit sphere, R = 1, (14), we obtain:-

$$(6\text{-}8) \qquad d\xi'' = \frac{\csc 1''}{2 \pi G} \int_q f'(\psi) \cos A \, \Delta g \, d q$$

The expression for $d\eta''$ being the same with sin A replacing cos A.

When Δg is in milligals, and when G is taken as 979.8 $cm./sec.^2$, the term before the integral sign will be:

$$c'' = \frac{\csc 1''}{2\,\pi\,G} = 0''.0335$$

One procedure is to perform numerical integration summing up the effect of each square over the earth as indicated in the following formula:-

$$(6\text{-}9) \qquad d\xi'' = c''\,\Sigma\, f'(\psi)_q \cos A_q\, \Delta g_q\, q$$

where q is the area of a 1° by 1° or 5° by 5° square and depends upon its mean latitude. The subscript q, where occurring, signifies mean values of the quantity for the square in question.

The squares method adapts readily to solution by high speed electronic computing machine, but manual graphical solution, though more time consuming, is made easier by use of maps on polar stereographic projection. With this projection templates can be easily prepared for use on the large area, small scale maps required, since a feature of the projection is that all circles on the sphere project as circles on the map, so azimuth lines (which are great circles) and the circular rings (small circles) can be easily drawn. Necessary formulas for preparing such templates may be found in (13) p. 259-260, and in (31) p. 98-100.

The functions $f'(\psi)$, $f'(\psi)\sin\psi$, and $\int f'(\psi)\sin\psi\, d\psi$, have been computed by Sollins at close intervals for different angular distances ψ, from the computation point. These tables have been published by Sollins (29) and are very useful in the solution of the problem.

[illegible]

6.3 The Circle-Ring Method.

In the relatively close area around the computation point (within $\psi \approx 3^{\circ}$), where a detailed gravity anomaly map is available, formula (6-7) is used in the following form:-

(6-10)
$$d\xi'' = \frac{\csc 1''}{2\,\pi\,G} \int_{A_1}^{A_2} \cos d\,A \int_{\psi}^{\psi+d\psi} f'(\psi)\,\sin\psi\,\Delta g_{\psi}\,d\,\psi$$

$$= \frac{\csc 1''}{2\,\pi\,G} (\sin A_2 - \sin A_1) \int_{\psi}^{\psi+d\psi} f'(\psi)\,\sin\psi\,\Delta g_{\psi}\,d\,\psi$$

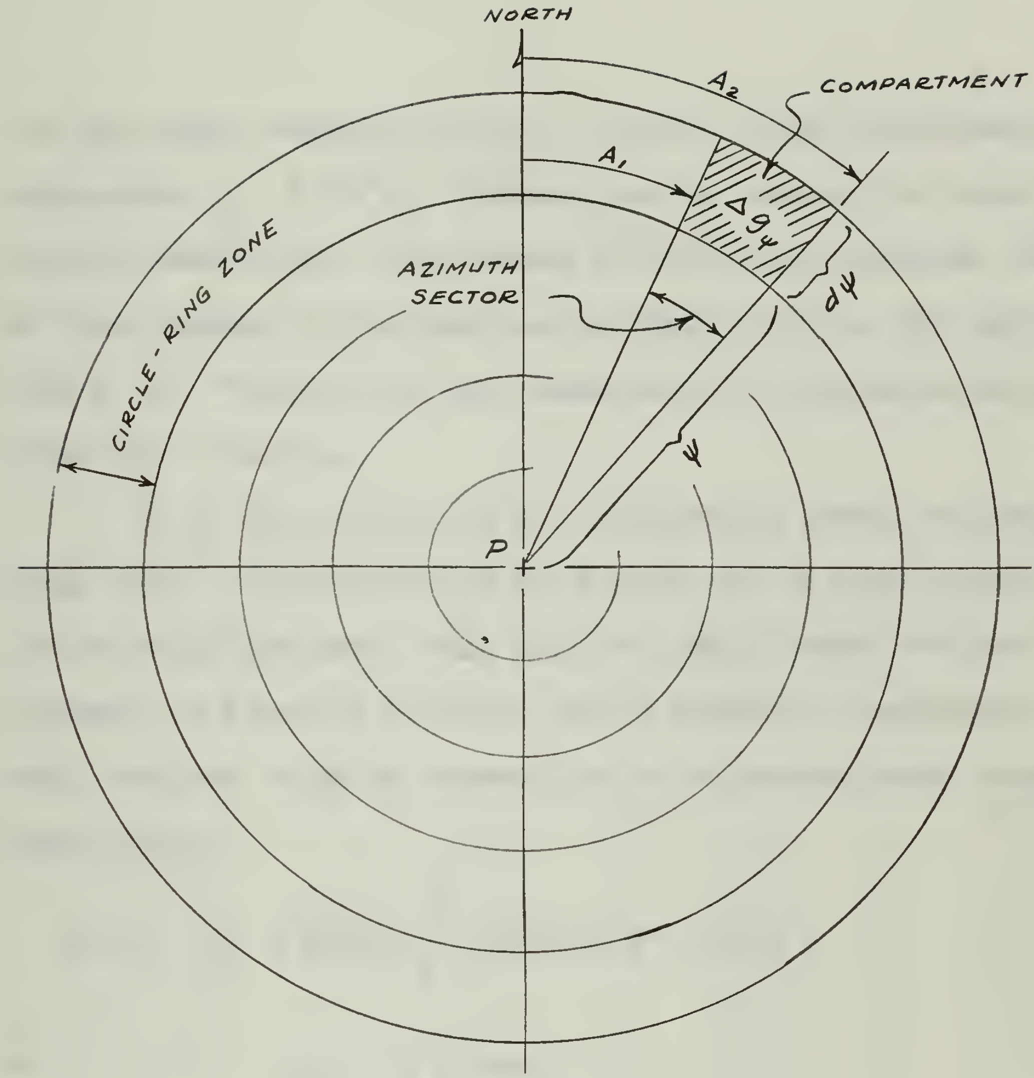
For $d\,\eta''$, substitute $(\cos A_1 - \cos A_2)$ for $(\sin A_2 - \sin A_1)$. Formula (6-10) gives the effect of a circle-ring compartment bounded by ψ and $\psi + d\psi$ and by A_1 and A_2, and having a mean anomaly Δg_{ψ}, on the total component ξ_g''.

Values of $\int_{4=10m}^{r} f'(\psi)\,\sin\psi\,d\,\psi$ are tabulated in Sollins Tables. Thus for the value of a particular $\int_{\psi}^{\psi+d\psi} f'(\psi)\,\sin\psi\,d\,\psi$, the tables would be used as indicated in the following expression:-

$$\int_{\psi=10m}^{\psi+d\psi} f'(\psi)\,\sin\psi\,d\,\psi - \int_{\psi=10m}^{\psi} f'(\psi)\,\sin\psi\,d\,\psi.$$

Figure 9 gives pictorial explanation of the components of the method and corresponding templates.

Any system of azimuth sector and ring zone intervals can be used in this method, but a system which yields equal effect for each compartment would obviate much unnecessary work. Kasansky developed a system in which the sine differences are 1/8 (32 compartments), and



P = computation point

ψ = inner radius of compartment

$\psi + d\psi$ = outer radius of compartment

A_1, A_2 = bounding azimuth lines

Δg_ψ = mean gravity anomaly in mgals, read from iso-anomaly lines on map

Fig. 9. Components of Template for Circle-Ring Method

the ring radii, beginning with $r_o = 5.0$ km., follow the geometrical progression $r_n = 1.270^n r_o$. Farther from the station, the ratio is slightly smaller than 1.270 because of the earth's curvature. Tables of these Kasansky Circle-Rings may be found in (13) p. 267, and in (16) p. 17. The effect of each compartment is 0.″001 radial deflection effect per 1 mgal Δg.

D. A. Rice, of U.S.C.& G.S., published a system of Circle-Rings (25), in which each ring was divided into 36 equal azimuth sectors of 10° aperture. Here also, the radial effect for each compartment for 1 mgal Δg is 0.″001. Rice's method for computation of the radii involved using the formula for the deflection effect along any radial line:-

$$\text{(6-11)} \quad d\theta'' = \frac{\csc 1''}{2\,\pi\,G} \int_s f'(\psi) \sin\psi \, d\psi \, \Delta g \, dA$$

so

$$d\xi'' = d\,\theta'' \cos A$$

$$d\eta'' = d\,\theta'' \sin A, \text{ and}$$

$$\text{(6-12)} \quad d\theta'' = \frac{\csc 1''}{2\,\pi\,G} \, \Delta g \, I_{\psi_o}^{\psi_1} \, \Delta A$$

where $I_{\psi_o}^{\psi_1}$ is obtained from Sollins' Tables (29).

Then for $\Delta A = 10^\circ = 0.17452$ radians, and for $G = 981{,}000$ mgals, formula (6-11) becomes:-

$$d\theta'' = 0.0058405484 \; I_{\psi_o}^{\psi_1} \, \Delta g.$$

For $d\theta'' = 0.''001$ and $\Delta g = 1$ mgal,

$$\Delta I_1 = I_{\psi_o}^{\psi_1} \qquad = 0.17121680$$

$$\Delta I_2 = I_{\psi_o}^{\psi_2} - I_{\psi_o}^{\psi_1} = 0.17121680, \text{ etc.}$$

Then selecting interpolated values of ψ from Sollins' table giving ΔI, the circle-ring table was prepared. Table I is Rice's table reproduced.

Table I

Rice Template Radii for Gravimetric Deflection of the Vertical

For radial deflection effect of 0.″001 and mean compartment anomaly of 1 mgal; angular aperture 10°.
(n = zone number; r = inner radius in km.)

n	r	n	r	n	r	n	r
1	0.119	14	1.099	27	10.15	39	77.97
2	0.141	15	1.304	28	12.05	40	92.22
3	0.167	16	1.547	29	14.29	41	109.0
4	0.198	17	1.836	30	16.94	42	128.7
5	0.235	18	2.179	31	20.09	43	151.9
6	0.279	19	2.586	32	23.83	44	179.1
7	0.331	20	3.068	33	28.25	45	210.9
8	0.393	21	3.641	34	33.48	46	248.0
9	0.467	22	4.320	35	39.67	47	291.2
10	0.554	23	5.125	36	47.00	48	341.2
11	0.657	24	6.081	37	55.66	49	399.0
12	0.780	25	7.216	38	65.90	50	465.5
13	0.926	26	8.560				

With the use of circle rings such as Rice's, the problem of computing the gravimetric deflections becomes simple, though laborious. All that is necessary is to estimate the mean anomaly from the contours in each compartment, make summation of Δg for each azimuth sector, multiply by 0.001 cos A_m or 0.001 sin A_m, and the effect for that sector is solved. Then the effects of all 36 sectors are added together for total effect. This may be further simplified when one takes note of

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the fact that mean azimuths in 180° opposite sectors will have the same cosine or sine, but with opposite sign. One must be careful with signs, since both the azimuth of the sector and the anomaly gradient contribute to the sign. Additional information on this aspect of the problem will be given in the explanation accompanying the computations made in Chapter 7.

In the event gravity material is poor, which is too often the case (and one has to be a good judge concerning when poor material is to be considered good), Heiskanen (13) p. 268, recommends combining two zones and two sectors into one four-compartment compartment, the effect of which then would be 0".004 per mgal.

6.4 The Inner Circle.

The circle-ring method, although it can be applied beyond $\psi = 3^\circ$, should not be used very close to the computation point since the term $\csc \frac{1}{2} \psi$ in the value of $f'(\psi) \sin \psi$, in formula (6-10) approaches infinity. Rice (25), suggests a lower limit of 100 meters. However this is usually too small because of the sparsity of gravity anomaly values, except in isolated instances, within any given 100 meter circle.

The inner circle, being small, is considered to have a uniform anomaly gradient, so the anomalies read from the cardinal points are used in the following expressions, the derivation of which may be found in (31) p. 93, 100-101:-

$$(6\text{-}13) \qquad d\xi'' = \frac{\csc 1''}{2\,G} \; \frac{\delta\,(\Delta g)}{\delta\,x} \times r_o = 0''.105\; r_o \,\frac{\delta\,(\Delta g)\,x}{\delta\,x}$$

$$d\eta'' = \frac{\csc 1''}{2\ G} \ \frac{\delta\ (\Delta g)\ y}{\delta\ y}\ r_o = 0''.105\ r_o\ \frac{\delta\ (\Delta g) y}{\delta\ y}$$

where r_o = radius of inner circle in km.

$\delta\ (\Delta g)_x = \Delta g_S - \Delta g_N$ in mgals (N. hemis.)

$\delta\ (\Delta g)_y = \Delta g_W - \Delta g_E$ in mgals (European practice)

$\delta x = 2\ r_o, \qquad \delta y = 2\ r_o$

Substituting these where appropriate, formulas (6-13) become:-

$$(6\text{-}14) \qquad d\xi'' = 0''.0525\ (\Delta g_S - \Delta g_N)$$

$$d\eta'' = 0''.0525\ (\Delta g_W - \Delta g_E)$$

See figure 10 for further clarification of this "one-gradient" method of computing the effect of the inner circle.

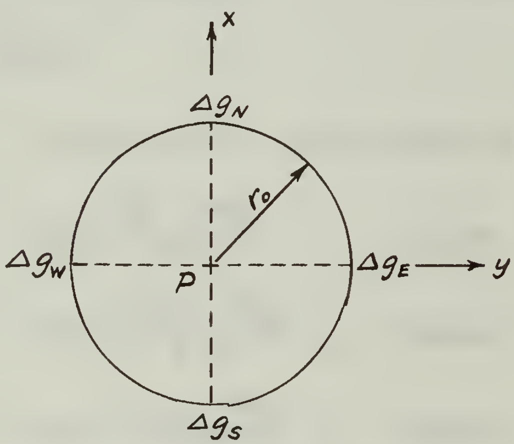

Fig. 10. Inner Circle - One-Gradient Method

Rice used a three-gradient method rather than one-gradient as above, in computations for 16 points in the U.S. (26). Taking anomalies at NE, SE, SW, and NW points, as well as at the cardinal

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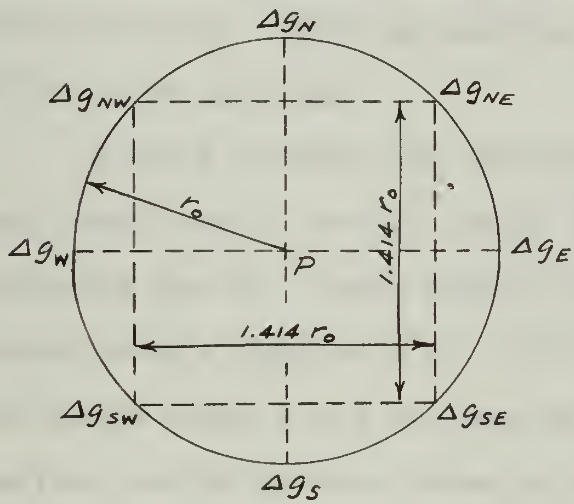
points of the circle, three N-S gradients and three E-W gradients were used, with weights of one-half given to the outer gradients and a weight of one to the center gradient. Figure 11 illustrates this. This method should be used only when detailed gravity information is available inside the inner circle.



$$\left.\begin{array}{l} \delta x \ (S-N) \\ \delta y \ (W-E) \end{array}\right\} = 2 r_o$$

$$\left.\begin{array}{ll} \delta x & (SW-NW) \\ & (SE-NE) \\ \delta y & (SW-SE) \\ & (NW-NE) \end{array}\right\} = 1.414 \, r_o$$

Fig. 11. Inner Circle - Three-Gradient Method

$$d\xi''(S-N) = 0''.105 \, \frac{\Delta g_s - \Delta g_N}{\delta x} \, r_o \qquad \text{Weight} = 1$$

$$d\xi''(SE-NE) = 0''.105 \, \frac{\Delta g_{sE} - \Delta g_{NE}}{\delta x} \, r_o \qquad \text{Weight} = \frac{1}{2}$$

$$d\xi''(SW-NW) = 0''.105 \, \frac{\Delta g_{sw} - \Delta g_{Nw}}{\delta x} \, r_o \qquad \text{Weight} = \frac{1}{2}$$

$$\Sigma \text{ Weights} = 2$$

Combining these, using weights shown and values of δx given in fig. 11, we obtain the following expression, given in (31) p. 101:-

$$(6\text{-}15) \qquad d\xi'' = 0.0262 \, (\Delta g_s - \Delta g_N) + 0.0186 \, (\Delta g_{SE} - \Delta g_{NE} + \Delta g_{sw} - \Delta g_{Nw})$$

This is for the northern hemisphere. For the southern hemisphere, just reverse S and N where occurring. The other component of the deflection is similar:-

$$(6\text{-}16)\quad d\eta'' = 0.0262\ (\Delta g_W - \Delta g_E) + 0.0186\ (\Delta g_{SW} - \Delta g_{SE} + \Delta g_{NW} - \Delta g_{NE})$$

This is for European practice, in which λ and η are considered positive eastward. Where the practice is opposite, W and E should be reversed where occurring.

As may be deduced from the opening paragraph, the smaller the inner circle, down to certain limits, the more concentrated the gravity information must be. Vening Meinesz (13) p. 70, states that the error incurred using a circle with r_o = 30 km. is less than 1%. A good round number seems to be 5 km., and for good results there should be from four to eight points of known Δg within the circle (13) p. 267.

Chapter 7. Computations

7.1 Dimensions from the 98th Meridian in the United States. (From Astro-Geodetic Deflections).

Using data from U.S.C. & G.S. Special Publication No. 229, (7), consisting of astronomic and geodetic latitudes and meridian components of the astro-geodetic deflections of the vertical for 30 points between 97° W and 99° W longitude, extending from North Dakota to Texas, values for a and f will be determined. The astro-geodetic deflections as published in the publication named have been reduced according to the Hayford isostatic assumption for a depth of compensation of 113.7 km.

The problem to be solved is $\Sigma \xi^2$ = minimum, according to the theory of least squares. The Vening Meinesz displacement and change of ellipsoid formulas will be used. To adapt them to practical use, terms containing the same unknowns must be collected and combined.

Prior to solution, a test problem was solved to determine what terms may be disregarded and whether or not the long or short formulas should be used. The maximum difference in longitude is about 1°, and terms containing longitude were determined to make negligible contribution to the residual deflection. Also, for the arcs of limited length ($\Delta \varphi \approx 13\overset{\circ}{.}3$, in this case) encountered in this problem, the difference between residuals computed by long and short formulas, turns out to be negligible. Terms containing δN will also be

neglected to simplify the problem.

The Vening Meinesz formulas, given as (4-19) and (4-23) in section 4.24, are here repeated with terms omitted as noted above.

Displacement:- $d\xi_D = \cos(\varphi_n - \varphi_o)\,\delta\xi_o$

Change of Ellipsoid:- $d\xi_c = p_1\delta\beta - p_2\delta f - p_3 f\,\delta f$

$$\text{where } \delta\beta = \frac{\delta a}{a} + \sin^2\varphi_o\,\delta f$$

Substituting the foregoing expression for $\delta\beta$ in $d\xi_c$ and collecting terms, we obtain:-

$$d\xi_c = p_1\frac{\delta a}{a} + p_1\sin^2\varphi_o\delta f - p_2\delta f - p_3 f\delta f$$

$$= \frac{p_1}{a}\delta a + [p_1\sin^2\varphi_o - p_2 - p_3 f]\,\delta f$$

$$\text{in which } p_1 = \sin(\varphi_n - \varphi_o)$$

$$p_2 = 4\cos\varphi_n\cos\tfrac{1}{2}(\varphi_n+\varphi_o)\sin\tfrac{1}{2}(\varphi_n - \varphi_o)$$

$$p_3 = (2 + \tfrac{3}{4}\tan\varphi_o\sin 4\varphi_o)\sin(\varphi_n - \varphi_o)$$

The observation equations will take the form given below, with subscript 0 signifying the initial point, Meades Ranch, Kansas, and subscript n signifying any other point in the arc.

$$\xi_n = d\xi_D + d\xi_c + (\varphi_n' - \varphi_n)$$

For convenience we transform the unknowns and the coefficients as indicated here:-

$$\xi_n = (h)\delta\xi_o + (j)\frac{\delta a}{1000} + (k)\,10{,}000\,\delta f + (l)$$

$$= hx + jy + kz + l$$

$$\left.\begin{aligned} \text{where} \quad x &= \delta\xi_o \\ y &= \frac{\delta a}{1000} \\ z &= 10{,}000\ \delta f \end{aligned}\right\}\ \text{unknowns.}$$

and
$$h = \cos\left(\varphi_n - \varphi_o\right)$$

$$j = \sin\left(\varphi_n - \varphi_o\right) \frac{1000 \csc 1''}{a}$$

$$k = \Big[\sin\left(\varphi_n - \varphi_o\right)[\ \sin^2 \varphi_o - \left(2 + \tfrac{3}{4} \tan \varphi_o \sin 4\ \varphi_o\right) f]$$

$$- 4 \cos \varphi_n \cos \tfrac{1}{2}\left(\varphi_n + \varphi_o\right) \sin \tfrac{1}{2}\left(\varphi_n - \varphi_o\right)\Big] \frac{\csc 1''}{10{,}000}$$

Table II - Basic Data - 98th Meridian

Point n	Location	Latitude φ_n	Astro-Geodetic Deflection $(\varphi' - \varphi)$
496	N.Dak.	46°02′17″N.	1″.17
571	S.Dak.	45 28 16	1.06
566	"	44 53 53	0.78
569	"	44 00 52	-2.93
570	"	43 43 24	-1.43
567	"	43 18 48	0.96
412	Nebr.	42 25 29	3.88
407	"	41 35 44	-1.29
413	"	40 46 35	-2.93
235	Kans.	39 46 35	-2.03
Meades R.	"	39 13 27	0.00
238	"	38 43 47	-1.29
245	"	37 51 38	-2.51
240	"	37 14 28	-0.40
522	Okla.	36 40 08	0.20
524	"	35 56 49	1.88
516	"	35 16 27	1.71
521	"	34 38 53	1.98
602	Texas	33 37 21	-1.28
620	"	32 48 10	-0.91
639	"	32 15 45	6.78
605	"	31 39 26	2.20
594	"	30 59 46	7.65
592	"	30 16 21	-1.11
635	"	29 42 49	-4.05
652	"	29 22 16	2.14
627	"	28 52 38	-0.59
634	"	28 18 06	-0.68
588	"	27 44 35	1.44
630	"	27 40 16	4.91

Clarke's 1866 Ellipsoid:-

$$a = 6{,}378{,}206 \text{ m.}$$
$$f = 1/294.98$$
$$= 0.00339006$$

Initial Point:- Meades Ranch

$$\varphi_o = 39°13'26''.686 \text{ N.}$$

[illegible]

Table III - Observation Equations - 98th Meridian

Point n	h (x)	j (y)	k (z)	$=(\varphi'_n - \varphi_n)$
496	0.993	3.837	-1.544	1.17
571	.994	3.519	-1.450	1.06
566	.995	3.197	-1.348	0.78
569	.997	2.701	-1.178	-2.93
570	.997	2.535	-1.118	-1.43
567	.997	2.306	-1.033	0.96
412	.998	1.806	-0.835	3.88
407	.999	1.338	- .637	-1.29
413	1.000	0.876	- .429	-2.93
235	1.000	.312	- .158	-2.03
Meades Ranch	1.000	--	--	0.00
238	1.000	- .279	.146	-1.29
245	1.000	- .770	.414	-2.51
240	.999	-1.119	.613	-0.40
522	.999	-1.442	.803	0.20
524	.998	-1.849	1.051	1.88
516	.998	-2.228	1.290	1.71
521	.997	-2.580	1.520	1.98
602	.995	-3.157	1.911	-1.28
620	.994	-3.617	2.367	-0.91
639	.993	-3.920	2.457	6.78
605	.991	-4.258	2.710	2.20
594	.990	-4.628	2.993	7.65
592	.988	-5.032	3.310	-1.11
635	.986	-5.353	3.567	-4.05
652	.985	-5.534	3.716	2.14
627	.984	-5.808	3.944	-0.59
634	.982	-6.128	4.214	-0.68
588	.980	-6.437	4.481	1.44
630	.980	-6.477	4.516	4.91

Table IV - Least Squares Solution - 98th Meridian

Forward Solution:-

	x	y	z	1	Σ
x	+29.62	-47.43	+35.76	+15.08	+33.03
	-1	+1.601283	-1.207292	-0.509115	-1.115125
y		+398.56	-250.22	-88.55	+12.36
		- 75.95	+ 57.26	+24.15	+52.89
(1)		+322.61	-192.96	-64.40	+65.25
		-1	+0.598122	+0.199622	-0.202257
z			+160.48	+58.88	+ 4.90
			- 43.17	-18.21	-39.88
			-115.41	-38.52	+39.03
( )			+ 1.90	+ 2.15	+ 4.05
			- 1	-1.131579	-2.131579
1				+223.62	+209.03
				- 7.68	- 16.82
				- 12.86	+ 13.03
				- 2.43	- 4.58
[ξξ]				+200.65	+200.66

Back Solution:-

	z	Check	y	Check	x	Check
1	-1.131579	-2.131579	+0.199622	-0.202257	-0.509115	-1.115125
z			-0.676822	-1.274944	+1.366146	+2.573438
y					-0.764132	-2.365417
Σ	-1.131579	-2.131579	-0.477200	-1.477201	+0.092899	-0.907104
Σ(check-U)		-1.000000		-1.000001		-1.000003

Standard Error of single observation:-

$$\mu = \pm \sqrt{\frac{[\xi\xi]}{n-u}} = \pm \sqrt{\frac{200.65}{30-3}} = \pm \sqrt{7.43148} = \pm 2''.726$$

[illegible]

Table V - Computation of Correlate Numbers - 98th Meridian

	1(x)	1(y)	1(z)
(x)	+ 1		
(y)	0 +1.6013		
	+1.6013	+ 1	
(z)	0 -1.2073 +0.9578	0 0 +0.5981	
	-0.2495	+0.5981	+ 1

$$1.90\ Q_{zz} = 1 \qquad 1.90\ Q_{yz} = 0.5981 \qquad 1.90\ Q_{xz} = -0.2495$$

$$Q_{zz} = 0.5263 \qquad Q_{yz} = 0.3148 \qquad Q_{xz} = -0.1313$$

$$322.61\ Q_{yy} - 192.96\ Q_{yz} = 1$$

$$322.61\ Q_{yy} = 1 + 60.7438 = 61.7438$$

$$Q_{yy} = 0.1914$$

$$322.61\ Q_{yx} - 192.96\ Q_{zx} = 1.6013$$

$$322.61\ Q_{yx} = 1.6013 - 25.3356 = -\ 23.7343$$

$$Q_{yx} = -\ 0.07357$$

$$29.62\ Q_{xx} - 47.43\ Q_{xy} + 35.76\ Q_{xz} = 1$$

$$29.62\ Q_{xx} = 1 - 3.4894 + 4.6953 = 2.2059$$

$$Q_{xx} = 0.07447$$

Standard Errors of Unknowns after Adjustment:-

$$m_x = \mu \sqrt{Q_{xx}} = 2.726 \sqrt{0.07447} = 2.726\ (\pm\ 0.2729) = \pm\ 0.7439$$

$$m_y = \mu \sqrt{Q_{yy}} = 2.726 \sqrt{0.1914} = 2.726\ (\pm\ 0.4375) = \pm\ 1.1926$$

$$m_z = \mu \sqrt{Q_{zz}} = 2.726 \sqrt{0.5263} = 2.726\ (\pm\ 0.2294) = \pm\ 0.6253$$

Results of least squares solution:-

$$x = +\ 0.092899 \quad = \delta\xi_o$$

$$y = -\ 0.477200 \quad = \frac{\delta a}{1000}$$

$$z = -\ 1.131579 \quad = 10{,}000\ \delta f$$

$$\delta\xi_o = +\ 0''.09 \ \pm\ 0''.74$$

$$\delta a = 1000\ (-\ 0.4772) = -\ 477.2 \text{ m.}$$

$$\delta f = \frac{-\ 1.131579}{10{,}000} = -\ 0.000113158$$

$$a_o = 6{,}378{,}206 \text{ m.}$$

$$\delta a = \underline{\qquad -477}$$

$$\text{New } a = 6{,}377{,}729 \text{ m.} \quad \pm\ 1190 \text{ m.}$$

$$f_o = 0.003390060$$

$$\delta f = \underline{-0.000113158}$$

$$\text{New } f = 0.003276902 \ \pm\ 0.00006253$$

$$1/f = 304.9 \pm 5.4$$

The solution, as can be readily seen, does not conform to what is known to be the truth, and the standard errors resulting show that the precision of the solution is "somewhat" rough. This of course can be explained by noting that the length of the arc is quite short, and that the number of points used is small. Better results certainly could have been obtained if many points having greater longitude differences had been used, thus affording the benefit of the prime vertical components of the deflections. However, the method used is what is intended to be demonstrated here.

It could have been expected that the results obtained from the data used would be as they are. Differences in latitude between the initial point and other points in the arc apparently differ systematically. It is seen as a rough check of the standard error of the value of a obtained in the least squares solution, that if we take the average value of $(\varphi'-\varphi)$ for the five northernmost stations and of the five southernmost stations, the resulting difference between the two is $-8''.57$. For the arc as short as the one used this is high.

Taking the mean latitudes of each of the two groups of five points, we get $\Delta\varphi = 16^{\circ}26'$. Taking the length of $\Delta(\varphi'-\varphi)$ at 31 meters per second of arc, we have $-8''.57$ @ 31 m. = -266 meters. Using one radian as $57^{\circ}.3$, we can determine the proportional difference for an arc of the length of the earth's radius:-

$$\frac{57.3}{16.4}(-266) = -930 \text{ meters.}$$

This kind of rough calculation could tell us in advance approximately what results can be expected.

7.2 Dimensions from Sixteen Stations in South Central United States by the Astro-Gravimetric Method.

For an illustration of the astro-gravimetric method as described in section 4.3, the published data on 16 points in South Central U.S. (including Kansas, Missouri, Oklahoma, Arkansas, Texas, and Louisiana) as computed by Rice (26), was used.

In the referenced publication, all geodetic data were converted from Clarke's 1866 Ellipsoid to the International Ellipsoid. In addition, as a result of the investigations, Rice determined that several of the astronomic observations which had been published in (7) were wrong, and were consequently done over. Moreover, all latitude and longitude determinations were reduced to the FK_3 system of cataloging mean places of stars. The gravity anomalies used were free-air.

All the data needed for solution of the present problem was given in Tables VI and VII of ref. (26), namely the latitudes and longitudes of each station, as well as the astro-geodetic and the gravimetric deflections for each. Here again, as in the problem of section 7.1, the extent of the area covered, and the number of stations involved are inadequate for a meaningful solution to the problem, but for illustrative purposes, the results are gratifying.

Meade's Ranch, Kansas, was again chosen as the initial point. The Vening Meinesz abbreviated formulas were determined to be adequate and were used for both ξ and η. Since Meade's Ranch was one of the 16 points given, the values of $\delta\xi_o$ and $\delta\eta_o$ are known, thus reducing the

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

number of unknowns to two (δN_o again being neglected).

In adapting the formulas (4-19), (4-23), (4-20), and (4-24) to this problem, the terms were grouped in the logical manner for ease in computation of the coefficients as follow:-

$$\xi = d\xi_D + d\xi_c + (\varphi' - \varphi) - \xi_g$$

$$\xi = g_1 \delta\xi_o + h_1 \delta\eta_o + j_1 \frac{\delta a}{100} + k_1 \,(10{,}000)\, \delta f + (\varphi' - \varphi) - \xi_g$$

$$g_1 = + \cos(\varphi - \varphi_o)$$

$$h_1 = + \sin\varphi \sin(\lambda - \lambda_o)$$

$$j_1 = + [\sin(\varphi - \varphi_o) - 2\cos\varphi_o \sin\varphi \sin^2 \tfrac{1}{2}(\lambda - \lambda_o)] \frac{100}{a} \csc 1''$$

$$\begin{aligned} k_1 = + \Big[& [\sin(\varphi - \varphi_o) - 2\cos\varphi_o \sin\varphi \sin^2 \tfrac{1}{2}(\lambda - \lambda_o)] \sin^2\varphi_o \\ & - 4\cos\varphi \cos\tfrac{1}{2}(\varphi + \varphi_o) \sin\tfrac{1}{2}(\varphi - \varphi_o) \\ & - (2 + \tfrac{3}{4}\tan\varphi_o \sin 4\varphi_o) \sin(\varphi - \varphi_o) f \Big] \frac{\csc 1''}{10{,}000} \end{aligned}$$

$$\eta = d\eta_D + d\eta_c + (\lambda' - \lambda)\cos\varphi - \eta_g$$

$$\eta = g_2 \delta\xi_o + h_1 \delta\eta_o + j_2 \frac{\delta a}{100} + k_2 \,(10{,}000)\, \delta f + (\lambda' - \lambda)\cos\varphi - \eta g$$

$$g_2 = - \sin\varphi_o \sin(\lambda - \lambda_o)$$

$$h_2 = + 1$$

$$j_2 = - \cos\varphi_o \sin(\lambda - \lambda_o) \frac{100}{a} \csc 1''$$

$$\begin{aligned} k_2 = [& - \cos\varphi_o \sin(\lambda - \lambda_o) \sin^2\varphi_o \\ & + \tfrac{1}{4}\sin\varphi_o \sin 4\varphi_o \sin(\lambda - \lambda_o) f] \frac{\csc 1''}{10{,}000} \end{aligned}$$

Then the observation equations become:-

$$\xi = j_1 \frac{\delta a}{100} + k_1 \ (10{,}000) \ \delta f + l_1$$

where $l_1 = g_1 \ \delta\xi_o + h_1\delta \ \eta_o + (\xi_a - \xi_g)$

$$\xi_a = \varphi' - \varphi$$

$$\delta \ \xi_o = - (\xi_a - \xi_g)_o = (\xi_g - \xi_a)_o$$

$$\delta \ \eta_o = - (\eta_a - \eta_g)_o = (\eta_g - \eta_a)_o$$

$$\eta = j_2 \frac{\delta a}{100} + k_2 \ (10{,}000) \ \delta f + l_2$$

where $l_2 = g_2\delta\xi_o + h_2\delta \ \eta_o + (\eta_a - \eta_g)$

$$\eta_a = (\lambda' - \lambda) \cos \varphi$$

Table VI - Basic Data - Rice's 16 Points

Point n	Name	φ				λ			
1	Meades R.	39°	13'	27"	N.	98°	32'	31"	W.
2	Arcadia	37	38	08		94	35	47	
3	Dirks	37	25	40		99	16	21	
4	Twin	36	06	57		96	46	40	
5	Little Rock	34	44	51		92	16	25	
6	Bogue	33	43	05		90	53	17	
7	Ecore	33	36	20		92	54	25	
8	Polk	33	23	05		99	05	46	
9	Bartley	33	12	08		101	40	44	
10	Roby	32	44	36		100	22	46	
11	Lacasa	32	39	05		98	41	30	
12	Burns	32	35	28		93	14	20	
13	Sears	32	33	31		100	02	16	
14	Brooks	32	25	09		92	07	01	
15	Bynum	32	19	16		101	00	00	
16	Legion	32	16	19		90	06	07	

International Ellipsoid:- $a = 6,378,388$ m.

$$f = 1/297$$

$$= 0.003367003367$$

Initial Point:- Meades Ranch

$$\varphi_o = 39^\circ\ 13'\ 26''.686\ \text{N.}$$

$$\lambda = 98^\circ\ 32'\ 30''.506\ \text{W.}$$

$$\delta\ \xi_o = (\xi_g - \xi_a)_o = -\ (\xi_a - \xi_g)_o = -(-0''.90) = +\ 0''.90$$

$$\delta\ \eta_o = (\eta_g - \eta_a)_o = -\ (\eta_a - \eta_g)_o = -(-0''.30) = +\ 0''.30$$

Table VI (cont'd)

Point	(1) $g_1 \delta \xi_o$	(2) $h_1 \delta \eta_o$	(3) $\xi_a - \xi_g$	(4) l_1
	(0".90 g_1)	(0".30) h_1	"	" (1) + (2) + (3)
1	+ 0.90	0	- 0.90	0
2	+ .90	- 0.01	- .16	+ 0.73
3	+ .90	+ .00	- .98	- .08
4	+ .90	- .01	- .73	+ .16
5	+ .90	- .02	+ .66	+ 1.54
6	+ .90	- .02	+ .56	+ 1.44
7	+ .90	- .02	+ .07	+ .95
8	+ .90	+ .00	- .75	+ .15
9	+ .90	+ .01	- .89	+ .02
10	+ .89	+ .01	- .36	+ .54
11	+ .89	+ .00	- .25	+ .64
12	+ .89	- .01	+ .56	+ 1.44
13	+ .89	+ .00	- .50	+ .39
14	+ .89	- .02	- .06	+ .81
15	+ .89	+ .01	- .45	+ .45
16	+ .89	- .02	+ .44	+ 1.31

Point	$g_2 \delta \xi_o$	$h_2 \delta \eta_o$	$\eta_a - \eta_g$	l_2
	(0".90 g_2)	(0".30 h_2)	"	" (1) + (2) + (3)
1	0	+ 0.30	- 0.30	0
2	+ 0.04	+ .30	- 1.24	- 0.90
3	- .01	+ .30	- .34	- .05
4	+ .02	+ .30	- .61	- .29
5	+ .06	+ .30	0.00	+ .36
6	+ .08	+ .30	- .05	+ .33
7	+ .06	+ .30	+ .76	+ 1.12
8	- .01	+ .30	- .02	+ .27
9	- .03	+ .30	+ 2.26	+ 2.53
10	- .02	+ .30	+ .74	+ 1.02
11	- .00	+ .30	+ .85	+ 1.15
12	+ .05	+ .30	+ .25	+ .60
13	- .01	+ .30	+ 1.14	+ 1.43
14	+ .06	+ .30	+ .47	+ .83
15	- .02	+ .30	+ 1.88	+ 2.16
16	+ .08	+ .30	+ .27	+ .65

[illegible]

Table VII - Observation Equations - Rice's 16 Points

Point	j	k	l
ξ 1	0	0	0
2	- 0.093	+ 0.672	+ .73
3	- .102	+ .773	- .08
4	- .176	+ 1.366	+ .16
5	- .261	+ 1.993	+ 1.54
6	- .323	+ 2.484	+ 1.44
7	- .323	+ 2.554	+ .95
8	- .329	+ 2.681	+ .15
9	- .341	+ 2.767	+ .02
10	- .366	+ 3.001	+ .54
11	- .370	+ 3.050	+ .64
12	- .379	+ 3.065	+ 1.44
13	- .376	+ 3.096	+ .39
14	- .392	+ 3.146	+ .81
15	- .390	+ 3.215	+ .45
16	- .406	+ 3.205	+ 1.31
η 1	0	0	0
2	- 0.172	+ 0.440	- 0.90
3	+ .032	- .081	- .05
4	- .077	+ .197	- .29
5	- .275	+ .702	+ .36
6	- .334	+ .850	+ .33
7	- .246	+ .627	+ 1.12
8	+ .024	- .062	+ .27
9	+ .137	- .349	+ 2.53
10	+ .080	- .205	+ 1.02
11	+ .007	- .017	+ 1.15
12	- .232	+ .590	+ .60
13	+ .065	- .167	+ 1.43
14	- .280	+ .714	+ .83
15	+ .107	- .274	+ 2.16
16	- .368	+ .937	+ .65

Table VIII Least Squares Solution - Rice's 16 Points

	$\frac{\delta a}{100}$	10,000 δf	1	Σ	1 (a)	1(f)
$\frac{\delta a}{100}$	+ 2.17	- 14.21	- 3.62	- 15.66	1	
	- 1	+6.54839	+1.66820	+7.21659		
10,000 δf		+ 106.07	+ 27.95	+ 119.81	0	
		- 93.05	- 23.71	- 102.55	+6.548	
		+ 13.02	+ 4.24	+ 17.26	+6.548	1
		- 1	-0.32565	-1.32565		
1			+ 30.84	+ 55.17		
			- 6.04	- 26.12		
			- 1.38	- 5.62		
[vv]			+ 23.42	+ 23.43		

	10,000 δf	Check	$\frac{\delta a}{100}$	Check
1	-0.32565	-1.32565	+1.66820	+7.21639
10,000 δf			-2.13248	-8.68087
Σ	-0.32565	-1.32565	-0.46428	-1.46426

Standard Error of Single Observation:-

$$\mu = \sqrt{\frac{[vv]}{n - u}} = \sqrt{\frac{23.42}{30}} = 0''.88$$

$$Q_{af} = \frac{6.548}{13.02} = 0.503$$

$$(+ 2.17\, Q_{aa} - 14.21\,(0.503) - 1 = 0$$

$$Q_{aa} = \frac{8.148}{2.17} = 3.755$$

$$Q_{ff} = \frac{1}{13.02} = 0.077$$

Standard Errors of Unknowns:- $m = \mu \sqrt{Q}$

$$\frac{m_a}{100} = 0.8836 \sqrt{3.755} = 1.712, \qquad 10{,}000\, m_f = 0.8836\sqrt{0.077} = 0.24520$$

$$m_a = 171.2 \text{ m} \qquad m_f = 0.000024520$$

Results of Least Squares Solution:-

$$10{,}000\ \delta f = -\ 0.32565 \qquad \pm\ 0.24520$$

$$\delta f = -\ 0.000032565 \qquad \pm\ 0.000024520$$

$$f_o = \underline{0.003367003}$$

$$f = f_o + \delta f = 0.003334438 \qquad \pm\ 0.000024520$$

$$\frac{1}{f} = 299.9 \ \pm\ 2.2$$

$$\frac{\delta a}{100} = -\ 0.46428 \ \pm\ 1.71 \text{ meters}$$

$$\delta a = -\ 46 \ \pm\ 171$$

$$a_o = \underline{6{,}378{,}388}$$

$$a = a_o + \delta a = 6{,}378{,}342 \ \pm\ 171 \text{ meters}$$

Here, as in the problem of section 7.1, the standard errors of the corrections are almost as large as or larger than the corrections themselves. But the fact that they, as well as the corrections to f and a themselves, are relatively small, testifies as to the value of the method. However, it must be realized that the standard errors reflect only the precision of the least squares adjustment, assuming the given data has no error, which of course is not true.

7.3 Computation of the Flattening from Gravity Anomalies.

Given a set of isostatic gravity anomalies (Hayford, D = 113.7 km.) at 67 points at various random positions scattered over the earth, and computed on the basis of Helmert's gravity formula of 1901, we can demonstrate the solution of a problem of correcting the gravity formula and of obtaining a value for the flattening of the ellipsoid (13), p. 76-79, (14). The method is that of Chapter 5, but extended to include the longitude term of the formula.

Helmert's Formula:-

$$\gamma = 978.030\ (1 + 0.005302\ \sin^2\varphi + 0.000007\ \sin^2 2\varphi)$$

for which the β term yields the value of f = 1/298.2. See (13), p. 78.

The general form of the gravity formula, including the longitude term is:-

$$\gamma = \gamma_E[1 + \beta\ \sin^2\varphi + \epsilon\ \sin^2 2\varphi + r\ \cos^2\varphi\ \cos 2(\lambda - \lambda_o)]$$

$$= \gamma_E[1 + \beta\ \sin^2\varphi + \epsilon \sin^2 2\varphi + r\ \cos^2\varphi\ (\cos 2\lambda \cos 2\lambda_o + \sin 2\lambda\ \sin 2\lambda_o)]$$

Making corrections, we have $\gamma' = \gamma + \Delta\gamma$,

where $\Delta\gamma = x' + y'\sin^2\varphi + z'\cos^2\varphi\ \cos 2\lambda + u'\cos^2\varphi\ \sin 2\lambda$

x' =	unknown corr'n to	γ_E	in mgals
y' =	" " "	β	in mgals ($\div 1.022 \times 10^{-3}$)
z' =	" " "	$r \cos 2\lambda_o$	in mgals ($\div 1.022 \times 10^{-3}$)
u' =	" " "	$r \sin 2\lambda_o$	in mgals ($\div 1.022 \times 10^{-3}$)

Using $\Delta\gamma - \Delta g = -v$, we have

$$x' + y'\sin^2\varphi + z'\cos^2\varphi \cos 2\lambda + u'\cos^2\varphi \sin 2\lambda - \Delta g = -v$$

The observation equations will then be:-

$$a\,x' + b\,y' + c\,z' + d\,u' - l = -v$$

where
$$a = 1$$
$$b = \sin^2\varphi$$
$$c = \cos^2\varphi \cos 2\lambda$$
$$d = \cos^2\varphi \sin 2\lambda$$
$$l = \Delta g$$

The solution, according to the method of least squares will be Σv^2 = minimum.

Table IX - Isostatic Gravity Anomalies (Hayford, D = 113.7 km.)

Point	φ	λ	Δg	Point	φ	λ	Δg
1	79°.5	11°.5	+ 16	34	22°.5	36°.5	+ 26
2	79.5	17.5	+ 8	35	18.5	38.5	+ 48
3	76.5	15.5	- 26	36	14.5	- 17.5	+ 44
4	71.5	27.5	+ 20	37	13.5	42.5	+ 43
5	65.5	24.5	- 5	38	- 15.5	38.5	+ 27
6	64.5	11.5	+ 28	39	- 26.5	15.5	+ 34
7	62.5	17.5	- 20	40	50.5	120.5	- 1
8	60.5	22.5	+ 24	41	49.5	94.5	+ 9
9	58.5	38.5	+ 16	42	48.5	116.5	+ 10
10	57.5	40.5	+ 28	43	47.5	96.5	+ 19
11	55.5	13.5	+ 10	44	47.5	79.5	+ 6
12	54.5	39.5	- 11	45	46.5	123.5	- 5
13	54.5	55.5	+ 4	46	46.5	92.5	+ 58
14	53.5	29.5	+ 15	47	46.5	60.5	- 1
15	53.5	50.5	+ 23	48	45.5	102.5	+ 23
16	52.5	25.5	+ 15	49	44.5	110.5	+ 14
17	51.5	- 0.5	+ 9	50	44.5	74.5	+ 26
18	51.5	11.5	+ 36	51	43.5	94.5	+ 14
19	50.5	10.5	+ 32	52	43.5	74.5	- 16
20	48.5	16.5	+ 31	53	42.5	71.5	+ 1
21	47.5	39.5	+ 43	54	39.5	120.5	- 20
22	46.5	7.5	+ 20	55	39.5	84.5	- 11
23	46.5	34.5	+ 6	56	38.5	78.5	- 5
24	45.5	10.5	+ 30	57	36.5	87.5	+ 14
25	44.5	- 1.5	+ 6	58	35.5	105.5	+ 11
26	44.5	39.5	+ 31	59	34.5	79.5	+ 10
27	43.5	42.5	+ 43	60	31.5	94.5	- 4
28	40.5	44.5	+ 74	61	56.5	59.5	+ 55
29	39.5	53.5	+ 15	62	51.5	104.5	- 2
30	36.5	+ 6.5	+ 14	63	46.5	61.5	+ 14
31	35.5	14.5	+ 70	64	38.5	70.5	- 66
32	28.5	33.5	+ 17	65	23.5	79.5	+ 18
33	26.5	33.5	+ 31	66	22.5	77.5	+ 27
				67	11.5	78.5	- 40

Table X - Error Equations - Flattening and Correction to Gravity Formula from Gravity Anomalies

Point	a (x')	b (y')	c (z')	d (u')	l = - Δg
1	1	.9669	.0306	.0130	- 16
2	1	.9669	.0272	.0190	- 8
3	1	.9456	.0467	.0281	+ 26
4	1	.8993	.0577	.0825	- 20
5	1	.8281	.1128	.1298	+ 5
6	1	.8147	.1706	.0724	- 28
7	1	.7868	.1746	.1223	+ 20
8	1	.7576	.1714	.1714	- 24
9	1	.7269	.0614	.2660	- 16
10	1	.7113	.0452	.2851	- 28
11	1	.6791	.2858	.1456	- 10
12	1	.6628	.0643	.3310	+ 11
13	1	.6628	- .1209	.3148	- 4
14	1	.6463	.1822	.3033	- 15
15	1	.6463	- .0062	.3537	- 23
16	1	.6295	.2332	.2880	- 15
17	1	.6125	.3874	- .0068	- 9
18	1	.6125	.3567	.1514	- 36
19	1	.5954	.3778	.1450	- 32
20	1	.5610	.3682	.2391	- 31
21	1	.5436	.0871	.4480	- 43
22	1	.5262	.4577	.1226	- 20
23	1	.5262	.1698	.4424	- 6
24	1	.5088	.4586	.1761	- 30
25	1	.4913	.5081	- .0266	- 6
26	1	.4913	.0971	.4994	- 31
27	1	.4739	.0459	.5242	- 43
28	1	.4217	.0101	.5781	- 74
29	1	.4046	- .0726	.5909	- 15
30	1	.3538	.6297	.1454	- 14
31	1	.3372	.5796	.3213	- 70
32	1	.2277	.3017	.7109	- 17
33	1	.1991	.3129	.7372	- 31

Table X (cont'd)

Point	a (x')	b (y')	c (z')	d (u')	l = - Δg
34	1	.1465	.2496	.8163	- 26
35	1	.1007	.2023	.8763	- 48
36	1	.0627	.7678	- .5376	- 44
37	1	.0545	.0825	.9420	- 43
38	1	.0714	.2089	.9048	- 27
39	1	.1991	.6865	.4124	- 34
40	1	.5954	- .1962	- .3539	+ 1
41	1	.5782	- .4165	- .0660	- 9
42	1	.5610	- .2642	- .3506	- 10
43	1	.5436	- .4448	- .1027	- 19
44	1	.5436	- .4261	.1636	- 6
45	1	.5262	- .1852	- .4362	+ 5
46	1	.5262	- .4721	- .0413	- 58
47	1	.5262	- .2441	.4062	+ 1
48	1	.5088	- .4452	- .2076	- 23
49	1	.4913	- .3840	- .3338	- 1[illegible]
50	1	.4913	- .4361	.2620	- 26
51	1	.4739	- .5197	- .0823	- 14
52	1	.4739	- .4511	.2710	+ 16
53	1	.4564	- .4341	.3271	- 1
54	1	.4046	- .2886	- .5207	+ 20
55	1	.4046	- .5844	.1136	+ 11
56	1	.3875	- .5638	.2393	+ 5
57	1	.3538	- .6438	.0564	- 14
58	1	.3372	- .5681	- .3413	- 11
59	1	.3208	- .6340	.2434	- 10
60	1	.2730	- .7180	- .1137	+ 4
61	1	.6954	- .147[illegible]	.2664	- 55
62	1	.6125	- .3389	- .1879	+ 2
63	1	.5262	- .2581	.3975	- 14
64	1	.3875	- .4759	.3854	+ 66
65	1	.1590	- .7852	.3014	- 18
66	1	.1465	- .7736	.3607	- 27
67	1	.0398	- .8839	.3752	+ 40

Table XI - Least Squares Solution - Flattening and Correction to Gravity Formula from Gravity Anomalies

Forward Solution:-

	x'	y'	z'	u'	1	Σ
x'	+67	+ 33.20	- 4.17	+ 13.17	- 1033	- 923.80
	- 1	-0.4955	+0.0622	-0.1966	+ 15.4179	+ 13.7881
y'		+ 19.76	- 1.29	+ 4.87	- 460.99	- 404.45
		- 16.45	+ 2.07	- 6.53	+ 511.85	+ 457.74
(1)		+ 3.31	+ 0.78	- 1.66	+ 50.86	+ 53.29
		- 1	-0.2356	+0.5015	- 15.3656	- 16.0997
z'			+ 10.73	+ 0.94	- 183.53	- 177.32
			- 0.26	+ 0.82	- 64.25	- 57.46
			- 0.18	+ 0.39	- 11.98	- 12.55
(2)			+ 10.29	+ 2.15	- 259.76	- 247.33
			- 1	-0.2089	+ 25.2439	+ 24.0350
u'				+ 9.65	- 337.37	- 308.74
				- 2.59	+ 203.09	+ 181.62
				- 0.83	+ 25.51	+ 26.72
				- 0.45	+ 54.26	+ 51.67
(3)				+ 5.78	- 54.51	- 48.73
				- 1	+ 9.4308	+ 8.4308
1					+52,563	+50,548.11
					-15,926.69	-14,243.06
					- 781.49	- 818.83
					- 6557.36	- 6243.32
					- 514.07	- 459.56
[vv]					+28,783.39	+28,783.34

Back Solution:-

	u'	Check	z'	Check	y'	Check	x'	Check
1	+9.4308	+8.4308	+25.2439	+24.0350	-15.3656	-16.0997	+15.4179	+13.7881
u'			- 1.9701	- 1.7612	+ 4.7295	+ 4.2280	- 1.8541	- 1.6575
z'					- 5.4833	- 5.2477	+ 1.4476	+ 1.3854
y'							+ 7.9872	+ 8.4827
Σ	+9.4308	+8.4308	+23.2738	+22.2738	-16.1194	-17.1194	+22.9986	+21.9987

Table XII - Correlate Equations - Flattening, etc., from Δg

	x	y	z	u	l_x	l_y	l_z	l_u
x	+67 - 1	+ 33.20 -0.4955	- 4.17 +0.0622	+13.17 -0.1966	1			
y					0 -0.4955			
(1)		+ 3.31 - 1	+ 0.78 -0.2356	- 1.66 +0.5015	-0.4955	1		
z					0 +0.0622 +0.1167	0 0 -0.2356		
(2)			+10.29 - 1	+ 2.15 -0.2089	+0.1789	-0.2356	1	
u					0 -0.1966 -0.2485 -0.0374	0 0 +0.5015 +0.0492	0 0 0 -0.2089	
(3)				+ 5.78	-0.4825	+0.5507	-0.2089	1

Solution of Correlate Numbers:-

$$5.78\, Q_{xu} = -\, 0.4825$$

$$Q_{xu} = -\, 0.08478$$

$$10.29\, Q_{xz} = +\, 2.15\, (-\, .08478) = 0.1789$$

$$Q_{xz} = (0.1789 + 0.1823) \div 10.29 = 0.03510$$

$$3.31\, Q_{xy} + 0.78\, (.03510) - 1.66\, (-\, .08478) = -\, 0.4955$$

$$Q_{xy} = (-\, 0.4955 - 0.0274 - 0.1407) \div 3.31 = -\, 0.20048$$

$$67\, Q_{xx} + 33.20\, (-\, .20048) - 4.17\, (.03510) + 13.17\, (-\, .08478) = 1$$

$$Q_{xx} = (1 + 6.6559 + 0.1464 + 1.1166) \div 67 = 0.13312$$

$$5.78\, Q_{yu} = 0.5507$$

$$Q_{yu} = 0.09528$$

$$10.29\, Q_{yz} + 2.15\, (.09528) = -\, 0.2356$$

$$Q_{yz} = (-\, 0.2356 - 0.2049) \div 10.29 = -\, 0.04281$$

$$3.31\, Q_{yy} + 0.78\, (-\, .04281) - 1.66\, (.09528) = 1$$

$$Q_{yy} = (1 + 0.0334 + 0.1582) \div 3.31 = 0.36000$$

$$5.78\, Q_{zu} = -\, 0.2089$$

$$Q_{zu} = -\, 0.03614$$

$$10.29\, Q_{zz} + 2.15\, (-\, .03614) = 1$$

$$Q_{zz} = (1 + 0.0777) \div 10.29 = 0.10473$$

$$5.78\, Q_{uu} = 1$$

$$Q_{uu} = 0.17301$$

Standard Error of an observation of unit weight:-

$$\mu = \sqrt{\frac{[vv]}{n - u}} = \sqrt{\frac{28,783}{67 - 4}} = \sqrt{456.87} = \pm 21.37 \text{ mgal}$$

Standard Errors of the unknowns after adjustment:-

$$m'_x = \mu \sqrt{Q_{xx}} \text{, etc.}$$

$$m_x' = 21.37 \sqrt{0.1331} = \pm \ 7.78 \text{ (mgal.)}$$

$$m_y' = 21.37 \sqrt{0.3600} = \pm 12.82 \text{ (mgal.} \div 1.022 \times 10^{-3}\text{)}$$

$$m_z' = 21.37 \sqrt{0.1047} = \pm \ 6.92 \text{ (mgal.} \div 1.022 \times 10^{-3}\text{)}$$

$$m_u' = 21.37 \sqrt{0.1730} = \pm \ 8.89 \text{ (mgal.} \div 1.022 \times 10^{-3}\text{)}$$

The values for the unknowns, solved in Table XI, and their standard errors, are here converted to gals.:-

$$x = x' \times 10^{-3} = + 0.0230 \pm 0.0078 \text{ gals.}$$

$$y = y' \times (1.022) \times 10^{-6} = - 0.000016 \pm 0.000013 \text{ gals.}$$

$$z = z' \times (1.022) \times 10^{-6} = + 0.0000238 \pm 0.0000071 \text{ gals.}$$

$$u = u' \times (1.022) \times 10^{-6} = + 0.0000096 \pm 0.0000091 \text{ gals.}$$

The components of the longitude term are:-

$$r^2 = z^2 + u^2 = 0.0006586 \times 10^{-6} \text{ , } r = 0.0000257$$

$$\tan 2 \lambda_o = \frac{u'}{z'} = \frac{9.4308}{23.2738} = 0.4052$$

$$2 \lambda_o = 22^o, \quad \lambda_o = 11^o$$

Applying the computed corrections:-

$$\gamma_E = 978.030 + x = 978.053 \text{ gals.}$$

$$\beta = 0.005302 + y = 0.005286$$

$$\epsilon = 0.000007 \quad \text{(unchanged)}$$

and adding the longitude term:-

$$0.000026 \cos^2\varphi \cos 2 (\lambda - 11^0)$$

we obtain the corrected gravity formula:-

$$\gamma = 978.053 [1 + 0.005286 \sin^2\varphi + 0.000007 \sin^2 2 \varphi + 0.000026 \cos^2\varphi \cos 2 (\lambda - 11^o)].$$

Next, the flattening f is solved from the corrected value of β. At this point it should be pointed out that if the principal problem is to find the flattening, it would not be necessary to include solution of the longitude term. With slight difference the same result for the correction to β is obtained when only two unknowns exist in the normal equations. Witness, using the first reduced normal equation in Table XI:-

$$3.31 y' + 50.86 = 0$$

$$y' = - 50.86 \div 3.31 = - 15.4$$

$$x = - 15.4 \times 1.022 \times 10^{-6} = - 0.000016$$

$$\beta = 0.005302 - 0.000016 = 0.005286$$

Now using Clairaut's Formula, (5-1), we obtain f from the new value of β.

$$0.0052855 = \frac{5}{2} m - f - \frac{17}{14} m f$$

$$m = \frac{\omega^2 a}{\gamma_E} \qquad \omega = \frac{2 \pi}{86164} \sec^{-1}$$

$$m = \left(\frac{2 \pi}{86164}\right)^2 \frac{637,820,000 \text{ cm.}}{978.05 \text{ gal.}}$$

$$= \frac{39.478 \times 637.82}{7424.2 \times 978.05} = \frac{25,179.858}{7,261,124} = 0.0034678$$

$$0.0052855 = \frac{5}{2}\,(0.0034678) - f\,[1 + \frac{17}{14}\,(0.0034678)]$$

$$= 0.008670 \;-\; f\,(1 + 0.0042109)$$

$$f \;=\; \frac{0.008670 - 0.005286}{1.0042109}$$

$$= \frac{0.0033840}{1.0042109} \;=\; 0.0033698 \;\pm\; 0.0000083$$

$$1/f \;=\; 296.75 \;\pm\; 0.74$$

$$(\beta = 0.0052855 \;\pm\; 0.0000130$$

$$\therefore m_f \;=\; \pm\,\frac{337}{529}\,(0.0000131) \;=\; \pm\; 0.0000083)$$

Of course, the result reflects the fact that only a few values of Δg were used, and at scattered locations. The solution by Heiskanen, referred to in Chapter 5, shows that the use of a great many points results in a value of f equal to that obtained from Helmert's original formula. The results in the computation given here are surprising considering the fact that the points chosen were completely random with no thought given to even approximately even distribution.

7.4 Computation of Gravimetric Deflections

7.41 Procedure and General Information.

The procedure used for computation of the gravimetric deflections at the two points chosen, is that involving the use of the Rice Circle-Rings. The extent of readily available gravity material dictated that the area covered in both cases was to be within the circle-ring no. 49, having a radius of 399.0 km.

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

A template was prepared to the scale of the available anomaly maps, having circles of radii listed in Table I and with 10° sectors. The usual manner of orienting the template, which is to have zero azimuth bisect one of the sectors making mean azimuths of all sectors multiples of 10°, was not followed here because the writer's method of assuring proper sign determination dictated that each compartment have its own quadrant designation. Therefore the azimuth lines were numbered in multiples of 10°, and the mean sector azimuths are multiples of 5°.

For the point in the northern hemisphere, zero azimuth is north, and for the one in the southern hemisphere it is south. Following the usual practice for points in Europe, λ and η were considered positive eastward, so for both points, the azimuths were measured eastward from zero; thus they are measured clockwise in the northern hemisphere, and counterclockwise in the southern hemisphere. This will preserve signs automatically, using the same procedures for both. Figure 12 illustrates the azimuth convention used.

Remembering that the plumb line (below the surface) is pulled toward the mass surplus and thus in the direction of the more positive of the anomalies on opposite sides of the computation point, we can devise a system of assuring correct sign. Figure 13 illustrates this. ξ is considered positive toward the nearest pole, η is considered positive eastward. Table XIII gives criteria to be followed.

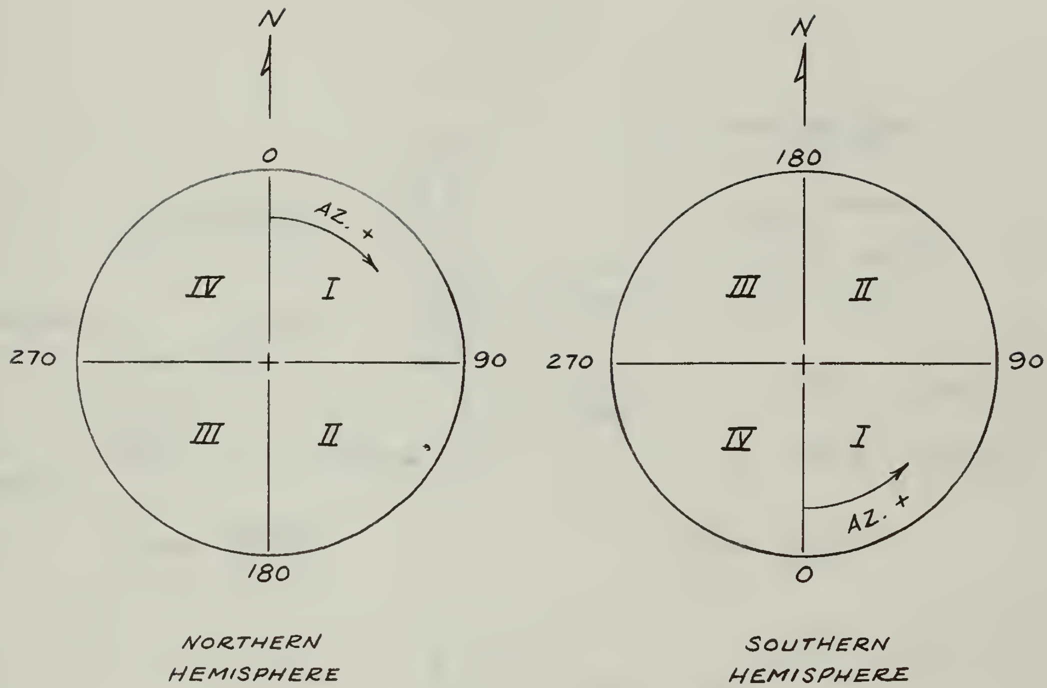


Fig. 12

Azimuth Convention for Computing Gravimetric Deflections for the Points Computed in this Thesis.

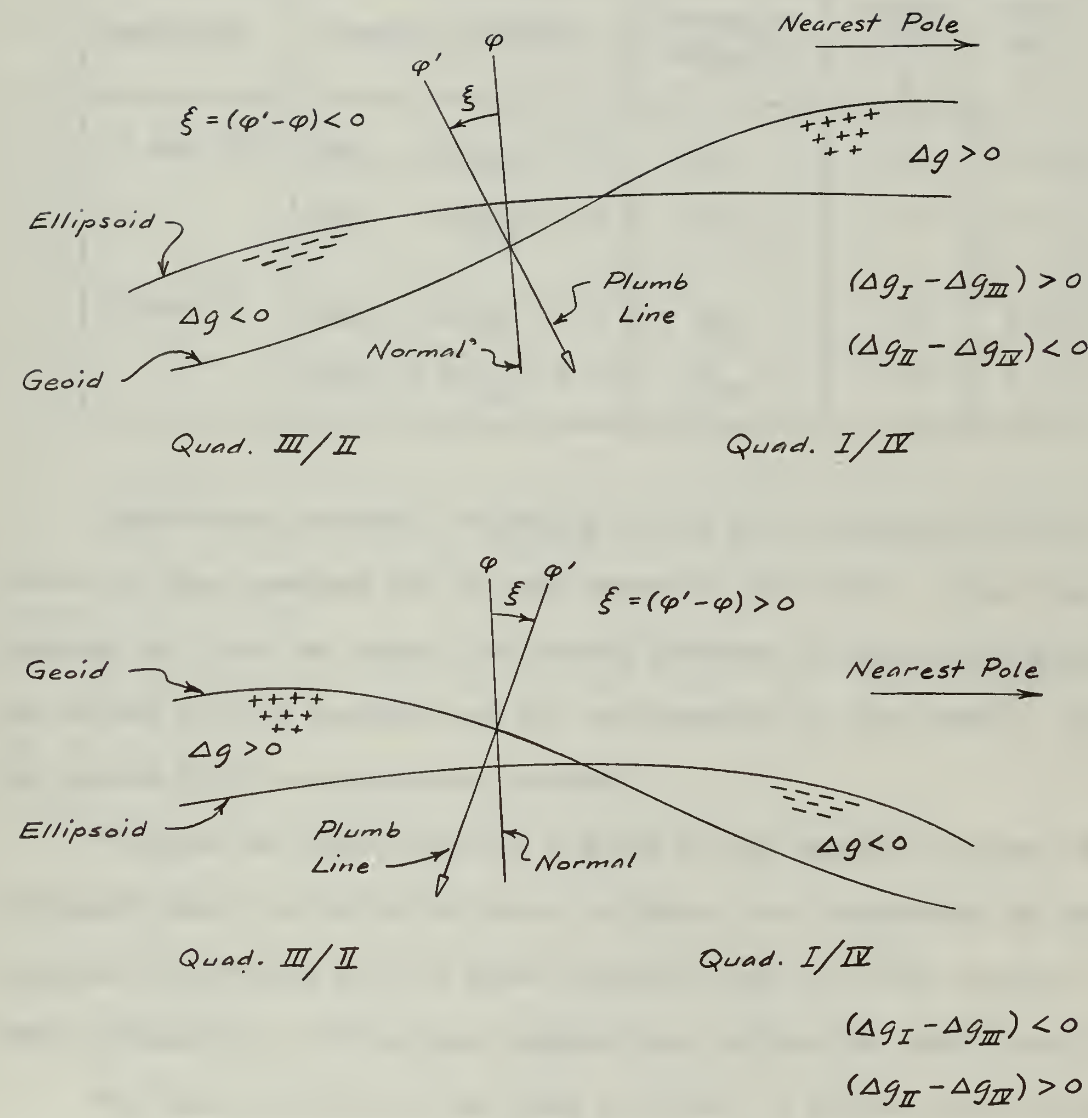

Nearest Pole
φ
φ'
ξ
$\xi = (\varphi' - \varphi) < 0$
$\Delta g > 0$
Ellipsoid
Plumb Line
$(\Delta g_I - \Delta g_{III}) > 0$
$\Delta g < 0$
$(\Delta g_{II} - \Delta g_{IV}) < 0$
Normal
Geoid
Quad. III/II
Quad. I/IV
φ
φ'
ξ
$\xi = (\varphi' - \varphi) > 0$
Geoid
Nearest Pole
$\Delta g > 0$
Ellipsoid
Plumb Line
Normal
$\Delta g < 0$
Quad. III/II
Quad. I/IV
$(\Delta g_I - \Delta g_{III}) < 0$
$(\Delta g_{II} - \Delta g_{IV}) > 0$

Fig. 13

Table XIII - Criteria for Determining Effective Azimuth and Sign of Deflection

Quadrants	Anomaly Gradient	Effective Azimuth	cosine and ξ	sine and η
I and III	$(\Delta g_{I} - \Delta g_{III}) < 0$	A_{I}	> 0	> 0
	$(\Delta g_{I} - \Delta g_{III}) > 0$	A_{III}	< 0	< 0
II and IV	$(\Delta g_{II} - \Delta g_{IV}) < 0$	A_{II}	< 0	> 0
	$(\Delta g_{II} - \Delta g_{IV}) > 0$	A_{IV}	> 0	< 0

Computation proceeds, beginning in the first quadrant with the effect of that quadrant and the one opposite, the third. After completing the first and third, the second quadrant is begun, determining the effect of that quadrant and the one opposite it, the fourth. Thus the entire 360° in azimuth are covered.

In order to obtain data for a graph of the effects of areas of different radii on the deflection of a point, the computation was made grouping the effects of four zones together from the point outward. Then accumulative effects were computed and plotted for both ξ and η.

The inner circle, for one case 5.125 km. in radius, and for the other 10.15 km., was computed by the one-gradient method. Insufficient local gravity data at the points dictated that the three-gradient method would yield results of meaningless accuracy.

The two points computed were Laiska, Finland (64° 03' N, 28° 50' E), and Wonderfontein, Union of South Africa (25° 48' S,

29° 53' E). The points were chosen to contribute data for the computation of the equatorial radius from the arc of the 30th meridian. In addition, their locations had to be such that the anomaly maps available to the writer would provide data to a large enough radius around each point to make computation by the circle-ring method feasible. The result was that both the points turned out to be somewhat removed from areas of concentrated gravity stations. However, point-anomaly values obtained from files of the Institute of Geodesy, Photogrammetry, and Cartography, The Ohio State University, for the general area around the points, allowed the plotting of iso-anomaly contour maps to the scale 1" = 6 km. (1 : 240,000) for both points, and the writer proceeded with the computation undaunted by the lack of enough gravity data.

The small scale maps used for the bulk of the computation were the following:-

Laiska : Free-air anomaly map of Finland on the scale 1 : 1,000,000, covering the area 21° - 35° East longitude (extending into Russia) and 60° - 69° North latitude. Contours for the Gulf of Bothnia were sketched in from a smaller scale map of that area. The maps used were prepared from gravity determinations of the Finnish Geodetic Institute and were computed on the International Gravity Formula. Contour interval was 5 mgal.

Wonderfontein : Bouguer anomaly lithographed map sheets of Union of South Africa, Northeast and Southeast sheets, on

scale 1 : 1,000,000, 10 mgal contours; total coverage of the two sheets : 22° 00' to 34° 30' S, 23° 30' to 33° 50' E. These maps were published in South Africa.

Topographic maps published as World Aeronautical Charts by U.S. Air Force (ACIC), scale 1 : 1,000,000; contour interval is 1000 ft., but adequate point elevations to the nearest foot are given for the use to which the maps were put. Total coverage of sheet numbers 1275, 1276, 1299, 1300, 1397, and 1398 is about the same as that of the Bouguer anomaly maps.

Since the South African point was estimated from a Bouguer anomaly map, it was necessary to use the topographic maps also, for the purpose of obtaining mean elevations for each compartment of the same Rice ring template used for the anomalies. Thus armed with the elevations, the Bouguer correction could be computed and subtracted from the Bouguer anomaly to obtain the Free Air anomaly for the compartment.

$$\Delta g_f = \Delta g_B - g'_B = \text{Free Air anomaly}$$

where Δg_B = Bouguer anomaly

$$g'_B = \text{Bouguer correction}$$

$$= -\frac{3}{4}\frac{\rho}{\rho_m}\left(2\,\frac{g}{R}\right) h$$

$$\rho = \text{density of crust assumed for area}$$

$$= 2.67$$

$$\rho_m = \text{mean density of earth}$$

$$= 5.576$$

$$\therefore g'_B = -\frac{3}{4}\,\frac{2.67}{5.576}\,(0.09406)\,h \text{ mgals} \qquad \text{(h in feet)}$$

$$= -0.03378\,h \text{ mgals.}$$

7.42 Computation for Laiska, Finland.

The computations for the circle-ring zones, proceeding from the outside zone, no. 48, and continuing inward toward the computation point to zone no. 27, are included in Table XIV.

The inner circle, r_o = 10.15 km., is computed below. The effects of both the inner circle and the circle-ring zones are accumulated in Table XV. Curves of the effect on deflections of in--creasing gravity field radius are shown in figure 15.

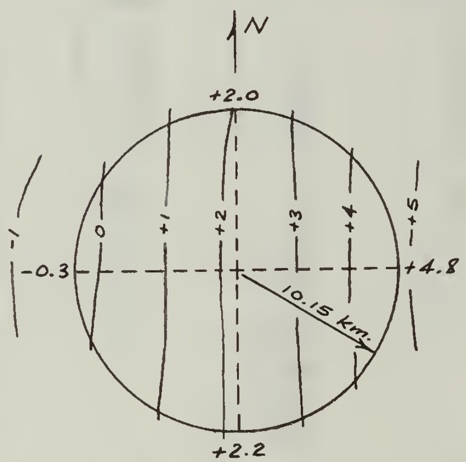


Fig. 14 - Inner Circle - Laiska.

Using the values shown in fig. 14, we obtain:-

$$d\xi'' = 0.0525\ (\Delta g_s - \Delta g_N)$$

$$= 0.0525\ (2.2 - 2.0) = 0.0525\ (0.2) = +\ 0''.010$$

$$d\eta'' = 0.0525\ (\Delta g_W - \Delta g_E)$$

$$= 0.0525\ (-\ 0.3 - 4.8) = 0.0525\ (-\ 5.1) = -\ 0''.268$$

Table XIV - Computation of Gravimetric Deflections at Laiska, Finland

Δg (mgal)

Quadrant Sector Zone	I 0 - 10	III 180-190	I 10 - 20	III 190-200	I 20 - 30	III 200-210
48	- 10	- 30	- 15	- 30	- 20	- 20
47	- 20	- 5	- 25	- 15	- 10	0
$\Sigma\ \Delta g$	- 30	- 35	- 40	- 45	- 30	- 20
I - III		+ 5		+ 5	- 10	
$d\theta$ @ 0".001		0".005		0".005	0".010	
Eff. A		185		195	25	
cos A		- .9962		- .9659	+ .9063	
sin A		- .0872		- .2588	+ .4226	
$d\xi$		- 0".005		- 0".005	+ 0".009	
$d\eta$		0".000		- 0".001	+ 0".004	
46	- 15	+ 15	- 10	+ 20	0	+ 20
45	0	+ 25	+ 5	+ 20	+ 5	+ 20
44	+ 10	+ 30	+ 10	+ 15	+ 10	+ 10
43	+ 10	+ 20	+ 10	+ 15	+ 15	+ 10
Σ	+ 5	+ 90	+ 15	+ 70	+ 30	+ 60
I - III	- 85		- 55		- 30	
$d\theta$	0.085		0.055		0.030	
A	5		15		25	
cos	+ .9962		+ .9659		+ .9063	
sin	+ .0872		+ .2588		+ .4226	
$d\xi$	+ .085		+ .053		+ .027	
$d\eta$	+ .007		+ .014		+ .013	
42	+ 15	+ 15	+ 15	+ 15	+ 18	+ 7
41	+ 18	+ 11	+ 20	+ 13	+ 22	+ 10
40	+ 18	+ 10	+ 20	+ 10	+ 25	+ 14
39	+ 15	+ 8	+ 20	+ 5	+ 24	+ 7
Σ	+ 66	+ 44	+ 75	+ 43	+ 89	+ 38
I - III		+ 22		+ 32		+ 51
$d\theta$		0.022		0.032		0.051
A		185		195		205
cos		- .9962		- .9659		- .9063
sin		- .0872		- .2588		- .4226
$d\xi$		- .022		- .031		- .046
$d\eta$		- .002		- .008		- .022

Table XIV (cont'd)

	I 30 - 40	III 210-220	I 40 - 50	III 220-230	I 50 - 60	III 230-240
48	+ 5	+ 10	+ 20	0	+ 35	- 10
47	+ 10	+ 10	+ 30	- 5	+ 45	- 10
Σ	+ 15	+ 20	+ 50	- 5	+ 80	- 20
I - III	- 5			+ 55		+ 100
dθ	.005			.055		.100
A	35			225		235
cos	+ .8192			- .7071		- .5736
sin	+ .5736			- .7071		- .8192
dξ	+ .004			- .039		- .057
dη	+ .003			- .039		- .082
46	+ 5	+ 5	+ 20	+ 5	+ 25	+ 10
45	+ 5	+ 5	+ 10	+ 10	+ 15	+ 5
44	+ 10	+ 10	+ 10	0	+ 10	0
43	+ 15	+ 5	+ 10	- 5	+ 10	0
Σ	+ 35	+ 25	+ 50	+ 10	+ 60	+ 15
I - III		+ 10		+ 40		+ 45
dθ		.010		.040		.045
A		215		225		235
cos		- .8192		- .7071		- .5736
sin		- .5736		- .7071		- .8192
dξ		- .008		- .028		- .026
dη		- .006		- .028		- .037
42	+ 18	+ 6	+ 17	+ 5	+ 14	+ 5
41	+ 22	+ 12	+ 22	+ 12	+ 17	+ 10
40	+ 27	+ 15	+ 26	+ 15	+ 20	+ 5
39	+ 34	+ 13	+ 28	+ 13	+ 20	+ 6
Σ	+ 101	+ 46	+ 93	+ 45	+ 71	+ 26
I - III		+ 55		+ 48		+ 45
dθ		.055		.048		.045
A		215		225		235
cos		- .8192		- .7071		- .5736
sin		- .5736		- .7071		- .8192
dξ		- .045		- .034		- .026
dη		- .032		- .034		- .037

Table XIV (cont'd)

	I 60 - 70	III 240-250	I 70 - 80	III 250-260	I 80 - 90	III 260-270
48	+ 45	- 15	+ 45	- 40	+ 35	- 40
47	+ 40	- 15	+ 35	- 25	+ 30	- 35
Σ	+ 85	- 30	+ 80	- 65	+ 65	- 75
I - III		+ 115		+ 145		+ 140
dθ		.115		.145		.140
A		245		255		265
cos		- .4226		- .2588		- .0872
sin		- .9063		- .9659		- .9962
dξ		- .049		- .038		- .012
dη		- .104		- .140		- .139
46	+ 15	- 10	+ 15	- 20	+ 20	- 25
45	+ 10	0	+ 10	- 15	+ 15	- 25
44	+ 10	+ 5	+ 10	- 10	+ 20	- 10
43	+ 10	- 5	+ 15	- 5	+ 20	- 5
Σ	+ 45	- 10	+ 50	- 50	+ 75	- 65
I - III		+ 55		+ 100		+ 140
dθ		.055		.100		.140
A		245		255		265
cos		- .4226		- .2588		- .0872
sin		- .9063		- .9659		- .9962
dξ		- .023		- .026		- .012
dη		- .050		- .097		- .139
42	+ 13	+ 5	+ 17	0	+ 15	- 3
41	+ 12	+ 5	+ 10	+ 3	+ 5	0
40	+ 10	0	+ 5	0	+ 4	0
39	+ 10	0	+ 6	0	+ 7	0
Σ	+ 45	+ 10	+ 38	+ 3	+ 31	- 3
I - III		+ 35		+ 35		+ 34
dθ		.035		.035		.034
A		245		255		265
cos		- .4226		- .2588		- .0872
sin		- .9063		- .9659		- .9962
dξ		- .015		- .009		- .003
dη		- .032		- .034		- .034

[illegible]

Table XIV (cont'd)

	II 90 - 100	IV 270-280	II 100-110	IV 280-290	II 110-120	IV 290-300
48	+ 30	- 15	+ 15	- 20	+ 10	- 20
47	+ 20	- 30	+ 10	- 30	+ 5	- 35
Σ	+ 50	- 45	+ 25	- 50	+ 15	- 55
II - IV		+ 95		+ 75		+ 70
dθ		.095		.075		.070
A		275		285		295
cos		+ .0872		+ .2588		+ .4226
sin		- .9962		- .9659		- .9063
dξ		+ .008		+ .019		+ .030
dη		- .095		- .072		- .063
46	+ 15	- 30	+ 5	- 35	0	- 25
45	+ 10	- 25	0	- 25	- 5	- 25
44	+ 15	- 5	0	- 5	- 10	- 20
43	+ 10	+ 5	- 5	+ 5	- 5	- 15
Σ	+ 50	- 55	0	- 60	- 20	- 85
II - IV		+ 105		+ 60		+ 65
dθ		.105		.060		.065
A		275		285		295
cos		+ .0872		+ .2588		+ .4226
sin		- .9962		- .9659		- .9063
dξ		+ .009		+ .016		+ .027
dη		- .105		- .058		- .059
42	0	- 2	0	- 7	0	- 5
41	+ 7	- 2	+ 12	- 7	+ 12	- 8
40	+ 15	- 2	+ 20	- 6	+ 18	- 10
39	+ 15	- 1	+ 20	- 7	+ 20	- 8
Σ	+ 37	- 7	+ 52	- 27	+ 50	- 31
II - IV		+ 44		+ 79		+ 81
dθ		.044		.079		.081
A		275		285		295
cos		+ .0872		+ .2588		+ .4226
sin		- .9962		- .9659		- .9063
dξ		+ .004		+ .020		+ .034
dη		- .044		- .076		- .073

Table XIV (cont'd)

	II 120-130	IV 300-310	II 130-140	IV 310-320	II 140-150	IV 320-330
48	+ 15	- 15	+ 15	- 20	+ 10	- 10
47	+ 10	- 20	+ 15	- 15	+ 15	- 10
Σ	+ 25	- 35	+ 30	- 35	+ 25	- 20
II - IV		+ 60		+ 65		+ 45
dθ		.060		.065		.045
A		305		315		325
cos		+ .5736		+ .7071		+ .8192
sin		- .8192		- .7071		- .5736
dξ		+ .034		+ .046		+ .037
dη		- .049		- .046		- .026
46	+ 5	- 25	+ 10	- 15	+ 15	- 15
45	- 5	- 25	+ 5	- 15	+ 15	- 10
44	- 5	- 20	0	- 5	+ 10	- 10
43	0	- 10	0	- 5	+ 10	- 10
Σ	- 5	- 80	+ 15	- 40	+ 45	- 45
II - IV		+ 75		+ 55		+ 90
dθ		.075		.055		.090
A		305		315		325
cos		+ .5736		+ .7071		+ .8192
sin		- .8192		- .7071		- .5736
dξ		+ .043		+ .039		+ .074
dη		- .061		- .039		- .052
42	+ 7	- 5	+ 5	- 5	+ 8	- 15
41	+ 13	- 8	+ 13	- 10	+ 10	- 20
40	+ 17	- 14	+ 16	- 14	+ 10	- 10
39	+ 20	- 8	+ 18	- 5	+ 10	- 8
Σ	+ 57	- 35	+ 52	- 34	+ 38	- 53
II - IV		+ 92		+ 86		+ 91
dθ		.092		.086		.091
A		305		315		325
cos		+ .5736		+ .7071		+ .8192
sin		- .8192		- .7071		- .5736
dξ		+ .053		+ .061		+ .075
dη		- .075		- .061		- .052

Table XIV (cont'd)

	II 150-160	IV 330-340	II 160-170	IV 340-350	II 170-180	IV 350-360
48	0	0	- 5	- 10	- 10	- 20
47	+ 10	- 5	+ 10	- 15	+ 10	- 15
Σ	+ 10	- 5	+ 5	- 25	0	- 35
II - IV		+ 15		+ 30		+ 35
dθ		.015		.030		.035
A		335		345		355
cos		+ .9063		+ .9659		+ .9962
sin		- .4226		- .2588		- .0872
dξ		+ .014		+ .029		+ .035
dη		- .006		- .008		- .003
46	+ 15	- 5	+ 15	- 10	+ 25	- 5
45	+ 15	- 10	+ 15	- 10	+ 25	0
44	+ 10	- 20	+ 10	- 10	+ 15	+ 10
43	+ 5	- 20	+ 5	0	+ 10	+ 15
Σ	+ 45	- 55	+ 45	- 30	+ 75	+ 20
II - IV		+ 100		+ 75		+ 55
dθ		.100		.075		.055
A		335		345		355
cos		+ .9063		+ .9659		+ .9962
sin		- .4226		- .2588		- .0872
dξ		+ .091		+ .072		+ .055
dη		- .042		- .019		- .005
42	+ 8	- 20	+ 7	- 10	+ 9	+ 12
41	+ 10	- 15	+ 10	- 13	+ 12	+ 5
40	+ 5	- 8	+ 10	- 5	+ 14	+ 4
39	+ 4	- 8	+ 3	- 2	+ 10	+ 5
Σ	+ 27	- 51	+ 30	- 30	+ 45	+ 26
II - IV		+ 78		+ 60		+ 19
dθ		.078		.060		.019
A		335		345		355
cos		+ .9063		+ .9659		+ .9962
sin		- .4226		- .2588		- .0872
dξ		+ .071		+ .058		+ .019
dη		- .033		- .016		- .002

[illegible]

Table XIV (cont'd)

	I 0 - 10	III 180-190	I 10 - 20	III 190-200	I 20 - 30	III 200-210
38	+ 17	+ 9	+ 29	+ 6	+ 24	+ 3
37	+ 16	+ 9	+ 25	+ 7	+ 20	+ 5
36	+ 18	+ 1	+ 24	+ 5	+ 19	+ 9
35	+ 21	0	+ 19	+ 1	+ 18	+ 6
Σ	+ 72	+ 19	+ 97	+ 18	+ 81	+ 23
I - III		+ 53		+ 79		+ 58
dθ		.053		.079		.058
A		185		195		205
cos		- .9962		- .9659		- .9063
sin		- .0872		- .2588		- .4226
dξ		- .053		- .076		- .053
dη		- .005		- .020		- .024
34	+ 16	0	+ 19	0	+ 18	+ 2
33	+ 12	0	+ 16	0	+ 17	+ 2
32	+ 10	+ 2	+ 13	+ 1	+ 16	+ 3
31	+ 8	+ 4	+ 12	+ 3	+ 16	+ 2
Σ	+ 46	+ 6	+ 60	+ 4	+ 67	+ 9
I - III		+ 40		+ 56		+ 58
dθ		.040		.056		.058
A		185		195		205
cos		- .9962		- .9659		- .9063
sin		- .0872		- .2588		- .4226
dξ		- .040		- .054		- .053
dη		- .003		- .014		- .025
30	+ 8	+ 5	+ 10	+ 4	+ 15	+ 3
29	+ 7	+ 5	+ 9	+ 4	+ 11	+ 3
28	+ 6	+ 4	+ 7	+ 3	+ 9	+ 3
27	+ 5	+ 4	+ 6	+ 3	+ 7	+ 3
Σ	+ 26	+ 18	+ 32	+ 14	+ 42	+ 12
I - III		+ 8		+ 18		+ 30
dθ		.008		.018		.030
A		185		195		205
cos		- .9962		- .9659		- .9063
sin		- .0872		- .2588		- .4226
dξ		- .008		- .017		- .027
dη		- .001		- .005		- .013

Table XIV (cont'd)

	I 30 - 40	III 210-220	I 40 - 50	III 220-230	I 50 - 60	III 230-240
38	+ 29	+ 7	+ 25	+ 8	+ 16	+ 5
37	+ 22	+ 5	+ 18	+ 3	+ 15	+ 1
36	+ 18	+ 5	+ 17	+ 2	+ 15	+ 1
35	+ 17	+ 6	+ 15	+ 5	+ 14	+ 5
Σ	+ 86	+ 23	+ 75	+ 18	+ 60	+ 12
I - III		+ 63		+ 57		+ 48
dθ		.063		.057		.048
A		215		225		235
cos		- .8192		- .7071		- .5736
sin		- .5736		- .7071		- .8192
dξ		- .052		- .040		- .028
dη		- .036		- .040		- .039
34	+ 15	+ 6	+ 14	+ 7	+ 13	+ 8
33	+ 15	+ 6	+ 14	+ 8	+ 14	+ 10
32	+ 17	+ 4	+ 18	+ 1	+ 19	+ 4
31	+ 19	+ 2	+ 20	+ 1	+ 20	0
Σ	+ 66	+ 18	+ 66	+ 17	+ 66	+ 22
I - III		+ 48		+ 49		+ 44
dθ		.048		.049		.044
A		215		225		235
cos		- .8192		- .7071		- .5736
sin		- .5736		- .7071		- .8192
dξ		- .039		- .035		- .025
dη		- .028		- .035		- .036
30	+ 16	+ 2	+ 16	+ 1	+ 17	+ 1
29	+ 12	+ 2	+ 12	+ 2	+ 13	+ 2
28	+ 9	+ 3	+ 9	+ 2	+ 8	+ 2
27	+ 7	+ 2	+ 7	+ 2	+ 6	+ 2
Σ	+ 44	+ 9	+ 44	+ 7	+ 44	+ 7
I - III		+ 35		+ 37		+ 37
dθ		.035		.037		.037
A		215		225		235
cos		- .8192		- .7071		- .5736
sin		- .5736		- .7071		- .8192
dξ		- .029		- .026		- .021
dη		- .020		- .026		- .030

Table XIV (cont'd)

	I 60 - 70	III 240-250	I 70 - 80	III 250-260	I 80 - 90	III 260-270
38	+ 11	0	+ 10	0	+ 10	+ 1
37	+ 13	0	+ 12	0	+ 13	+ 1
36	+ 13	+ 1	+ 11	+ 2	+ 14	+ 3
35	+ 12	+ 5	+ 10	+ 5	+ 15	+ 6
Σ	+ 49	+ 6	+ 43	+ 7	+ 52	+ 11
I - III		+ 43		+ 36		+ 41
dθ		.043		.036		.041
A		245		255		265
cos		- .4226		- .2588		- .0872
sin		- .9063		- .9659		- .9962
dξ		- .018		- .009		- .004
dη		- .039		- .035		- .041
34	+ 12	+ 9	+ 10	+ 9	+ 15	+ 6
33	+ 14	+ 10	+ 14	7	+ 15	+ 3
32	+ 19	+ 3	+ 20	+ 1	+ 18	0
31	+ 20	0	+ 19	0	+ 18	0
Σ	+ 65	+ 22	+ 63	+ 17	+ 66	+ 9
I - III		+ 43		+ 47		+ 57
dθ		.043		.047		.057
A		245		255		265
cos		- .4226		- .2588		- .0872
sin		- .9063		- .9659		- .9962
dξ		- .018		- .012		- .005
dη		- .039		- .045		- .057
30	+ 16	+ 1	+ 15	0	+ 14	0
29	+ 12	+ 1	+ 10	0	+ 10	0
28	+ 8	0	+ 7	0	+ 7	0
27	+ 6	0	+ 5	0	+ 5	0
Σ	+ 42	+ 2	+ 37	0	+ 36	0
I - III		+ 40		+ 37		+ 36
dθ		.040		.037		.036
A		245		255		265
cos		- .4226		- .2588		- .0872
sin		- .9063		- .9659		- .9962
dξ		- .017		- .010		- .003
dη		- .036		- .036		- .036

[illegible]

Table XIV (cont'd)

	II 90-100	IV 270-280	II 100-110	IV 280-290	II 110-120	IV 290-300
38	+ 15	- 3	+ 20	- 6	+ 20	- 5
37	+ 15	- 1	+ 20	- 5	+ 20	- 4
36	+ 17	+ 2	+ 20	- 2	+ 20	0
35	+ 18	+ 6	+ 20	+ 5	+ 20	+ 5
Σ	+ 65	+ 4	+ 80	- 8	+ 80	- 4
II - IV		+ 61		+ 88		+ 84
$d\theta$		.061		.088		.084
A		275		285		295
cos		+ .0872		+ .2588		+ .4226
sin		- .9962		- .9659		- .9063
$d\xi$		+ .006		+ .023		+ .035
$d\eta$		- .061		- .085		- .076
34	+ 20	+ 5	+ 20	+ 5	+ 20	+ 5
33	+ 19	+ 3	+ 20	+ 4	+ 20	+ 7
32	+ 18	+ 1	+ 18	+ 1	+ 18	+ 2
31	+ 16	0	+ 15	0	+ 15	0
Σ	+ 73	+ 9	+ 73	+ 10	+ 73	+ 14
II - IV		+ 64		+ 63		+ 59
$d\theta$		.064		.063		.059
A		275		285		295
cos		+ .0872		+ .2588		+ .4226
sin		- .9962		- .9659		- .9063
$d\xi$		+ .006		+ .016		+ .025
$d\eta$		- .064		- .061		- .054
30	+ 13	0	+ 12	0	+ 12	0
29	+ 10	0	+ 10	0	+ 9	0
28	+ 7	0	+ 7	0	+ 6	0
27	+ 5	0	+ 5	0	+ 5	0
Σ	+ 35	0	+ 34	0	+ 32	0
II - IV		+ 35		+ 34		+ 32
$d\theta$		.035		.034		.032
A		275		285		295
cos		+ .0872		+ .2588		+ .4226
sin		- .9962		- .9659		- .9063
$d\xi$		+ .003		+ .009		+ .014
$d\eta$		- .035		- .033		- .029

Table XIV (cont'd)

	II 120-130	IV 300-310	II 130-140	IV 310-320	II 140-150	IV 320-330
38	+ 20	0	+ 19	- 5	+ 10	- 8
37	+ 20	- 3	+ 19	- 6	+ 15	- 6
36	+ 20	- 1	+ 19	- 2	+ 16	- 3
35	+ 20	+ 2	+ 20	+ 1	+ 18	0
Σ	+ 80	- 2	+ 77	- 12	+ 59	- 17
II - IV		+ 82		+ 89		+ 76
dθ		.082		.082		.076
A		305		315		325
cos		+ .5736		+ .7071		+ .8192
sin		- .8192		- .7071		- .5736
dξ		+ .047		+ .063		+ .062
dη		- .067		- .063		- .044
34	+ 20	+ 5	+ 20	+ 3	+ 19	+ 2
33	+ 19	+ 8	+ 17	+ 3	+ 15	+ 2
32	+ 17	+ 2	+ 14	+ 2	+ 13	0
31	+ 14	0	+ 12	0	+ 11	0
Σ	+ 70	+ 15	+ 63	+ 8	+ 58	+ 4
II - IV		+ 55		+ 55		+ 54
dθ		.055		.055		.054
A		305		315		325
cos		+ .5736		+ .7071		+ .8192
sin		- .8192		- .7071		- .5736
dξ		+ .032		+ .039		+ .044
dη		- .045		- .039		- .031
30	+ 11	0	+ 11	0	+ 10	0
29	+ 9	0	+ 9	0	+ 8	0
28	+ 7	0	+ 7	0	+ 6	0
27	+ 5	0	+ 5	0	+ 5	0
Σ	+ 32	0	+ 32	0	+ 29	0
II - IV		+ 32		+ 32		+ 29
dθ		.032		.032		.029
A		305		315		325
cos		+ .5736		+ .7071		+ .8192
sin		- .8192		- .7071		- .5736
dξ		+ .018		+ .023		+ .024
dη		- .026		- .023		- .017

Table XIV (cont'd)

	II 150-160	IV 330-340	II 160-170	IV 340-350	II 170-180	IV 350-360
38	+ 4	- 2	+ 1	0	+ 5	+ 5
37	+ 7	- 4	+ 3	0	+ 4	+ 5
36	+ 10	- 3	+ 4	+ 2	+ 1	+ 8
35	+ 11	- 1	+ 1	+ 2	0	+ 10
Σ	+ 32	- 10	+ 9	+ 4	+ 10	+ 28
II - IV		+ 42		+ 5	- 18	
dθ		.042		.005	.018	
A		335		345	175	
cos		+ .9063		+ .9659	- .9962	
sin		- .4226		- .2588	+ .0872	
dξ		+ .038		+ .005	- .018	
dη		- .018		- .001	+ .002	
34	+ 12	+ 1	+ 5	+ 3	+ 1	+ 8
33	+ 15	+ 1	+ 10	+ 3	+ 3	+ 6
32	+ 13	+ 1	+ 9	+ 2	+ 6	+ 5
31	+ 10	0	+ 8	+ 2	+ 7	+ 5
Σ	+ 50	+ 3	+ 32	+ 10	+ 17	+ 24
II - IV		+ 47		+ 22	- 7	
dθ		.047		.022	.007	
A		335		345	175	
cos		+ .9063		+ .9659	- .9962	
sin		- .4226		- .2588	+ .0872	
dξ		+ .043		+ .021	- .007	
dη		- .020		- .006	+ .001	
30	+ 9	0	+ 8	+ 2	+ 6	+ 5
29	+ 8	0	+ 7	+ 3	+ 5	+ 5
28	+ 6	0	+ 5	+ 1	+ 5	+ 5
27	+ 5	0	+ 5	+ 1	+ 4	+ 3
Σ	+ 28	0	+ 25	+ 7	+ 20	+ 18
II - IV		+ 28		+ 18		+ 2
dθ		.028		.018		.002
A		335		345		355
cos		+ .9063		+ .9659		+ .9962
sin		- .4226		- .2588		- .0872
dξ		+ .025		+ .017		+ .002
dη		- .012		- .005		- .000

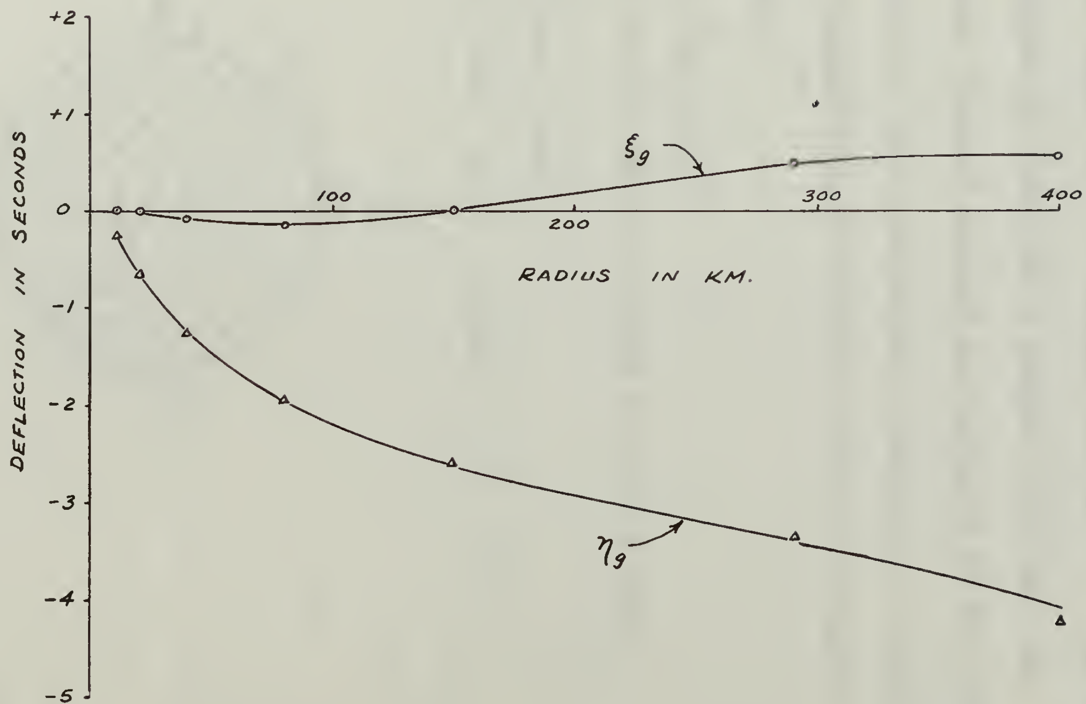
[illegible]

Table XV Summation of Effects of Inner Circle
and Zones 27 - 48 for Laiska

Zone Group	Group Sum $d\xi''$	Accumulative $d\xi''$	Group Sum $d\eta''$	Accumulative $d\eta''$	Outer Radius (km.)
Inner	+ 0.010	+ 0.010	- 0.268	- 0.268	10.15
27 - 30	- 0.023	- 0.013	- 0.383	- 0.651	20.09
31 - 34	- 0.062	- 0.075	- 0.601	- 1.252	39.67
35 - 38	- 0.072	- 0.147	- 0.692	- 1.944	77.97
39 - 42	+ 0.164	+ 0.017	- 0.667	- 2.611	151.9
43 - 46	+ 0.468	+ 0.485	- 0.763	- 3.374	291.2
47 - 48	+ 0.060	+ 0.545	- 0.866	- 4.240	399.0
Σ	ξ_g = + 0.545		η_g = - 4.240		

FIG. 15 - EFFECT ON DEFLECTIONS OF INCREASING GRAVITY FIELD RADIUS

STATION : LAISKA, FINLAND
64° 03′ N.
28° 50′ E.



7.43 Computation for Wonderfontein, Union of South Africa.

As mentioned in section 7.41, both Bouguer anomaly maps and topographic maps were required for this station. This situation prevailed from the outer zone, no. 48, through zone no. 31. The area within the circle described by that zone's radius, 20.09 km., was covered by a larger scale free-air anomaly map prepared by the writer.

The inner circle for this station had a radius r_o = 5.125 km. Figure 16, below is a reproduction of the inner circle. Using the values shown, we obtain:-

$$d\xi'' = 0.0525\ (\Delta g_N - \Delta g_s)$$
$$= 0.0525\ (99.0 - 88.5) = 0.0525\ (10.5) = +\ 0.''551$$

$$d\eta'' = 0.0525\ (\Delta g_W - \Delta g_E)$$
$$= 0.0525\ (88.2 - 90.2) = 0.0525\ (-2.0) = -0.''105$$

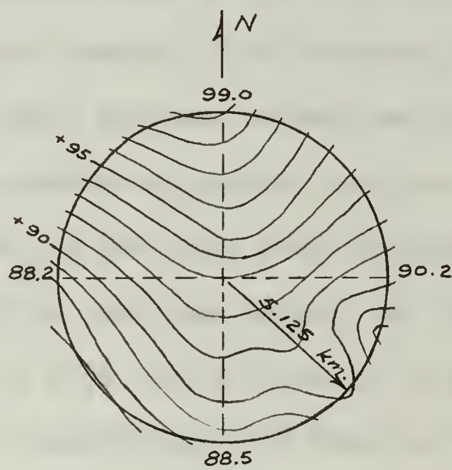


Fig. 16 - Inner Circle - Wonderfontein.

The effects of both the inner circle and the circle-ring zones are accumulated in Table XVI. Curves of the effect on deflections of increasing gravity field radius are shown in fig. 17.

7.44 Comments on the Results of Deflection Computation.

The results shown in Tables XV and XVI and in Figures 15 and 17, are the computed effects of the gravity field around each point out to a radius of 399.0 km. The effect of the rest of the world, beyond 399.0 km. should not be ignored in practice. Time and availability of data do not permit careful computation of this effect for purposes of this thesis. However, an idea of its magnitude can be gotten from publications containing results of investigations into this problem by various authors. For instance, Kaula (19), (20), determined the RMS uncertainties in the total deflection as computed from the Vening Meinesz formulas, first assuming perfect knowledge of gravity within given radii and none beyond, and second, ideal distribution of given numbers of gravity stations out to given radii. Taking Kaula's figures for the first case, assuming perfect knowledge within a radius of 200 km. and none beyond, the RMS uncertainty is 4\".1; for 500 km. it is 3\".3; for 1000 km. it is 2\".6; etc. It should be noted that Kaula's computations were based on information provided by Hirvonen (39).

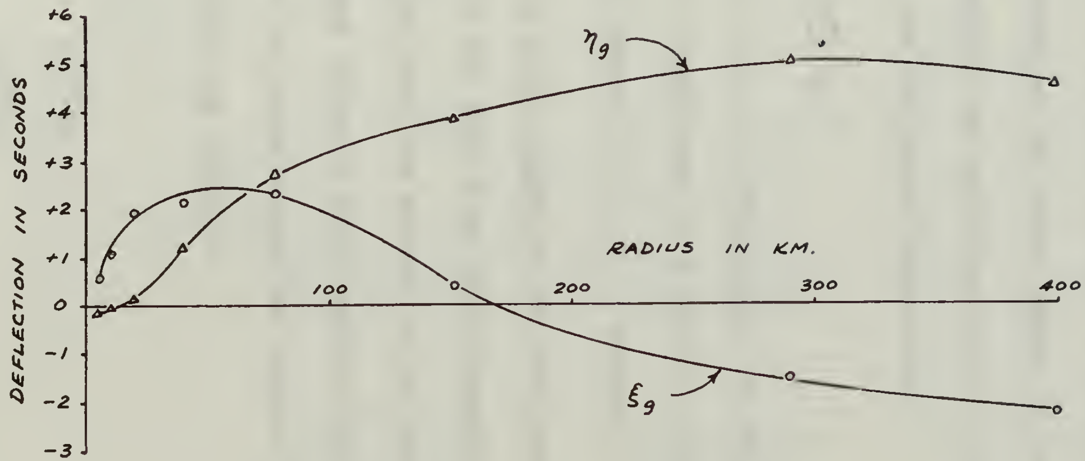
Interpolating for 399 km. radius, an RMS uncertainty of $\pm$ 3\".6 is obtained if Kaula's information is assumed to apply. The probable error is then $\pm$ 0.6745 (3\".6) = $\pm$ 2\".4.

Table XVI - Summation of Effects of Inner Circle and Zones 23 - 48 for Wonderfontein.

Zone Group	Group Sum $d\xi''$	Accumulative $d\xi''$	Group Sum $d\eta''$	Accumulative $d\eta''$	Outer Radius (km.)
Inner	+ 0.551	+ 0.551	- 0.105	- 0.105	5.125
23 - 26	+ 0.515	+ 1.066	+ 0.013	- 0.092	10.15
27 - 30	+ 0.905	+ 1.971	+ 0.212	+ 0.120	20.09
31 - 34	+ 0.128	+ 2.099	+ 1.099	+ 1.219	39.67
35 - 38	+ 0.175	+ 2.274	+ 1.448	+ 2.667	77.97
39 - 42	- 1.884	+ 0.390	+ 1.139	+ 3.806	151.9
43 - 46	- 1.919	- 1.529	+ 1.222	+ 5.028	291.2
47 - 48	- 0.754	- 2.283	- 0.493	+ 4.535	399.0
Σ	ξ_g = - 2.283		η_g = + 4.535		

FIG. 17 - EFFECT ON DEFLECTIONS OF INCREASING GRAVITY FIELD RADIUS

STATION : WONDERFONTEIN, UNION OF SOUTH AFRICA
25° 48' S.
29° 53' E.



Estimates of the probable errors of the gravimetric deflections as computed from the data given within the field of radius 399.0 km. must be made, since perfect knowledge of gravity within the field can not be assumed. The method to be used for this estimate is that of the U.S.C. & G.S. and is described in Rice's paper on the 16 points (26). According to this method, the probable error in either component ξ or η, is determined from the following expression:-

$$E_t^2 = 18\ \Sigma\ (0''.001\ E_r)^2$$

where E_r = Probable error in determining the mean anomaly in any compartment in a sector in mgals.

E_t = Probable error in ξ or η in seconds

In making use of the above equation for this problem, the probable error was estimated for each zone group computed. The accumulated results, combined with the effect of the inner circle, each component of which is assumed to be no more certain than its value, and with the probable error for the area beyond the radius 399.0 km., give the following (see tables XVII and XVIII):-

For <u>Total</u> Deflection θ :-

Laiska $E_t = \pm 2''.42$

Wonderfontein $E_t = \pm 2''.47$.

Thus it appears that for computations based upon the gravity data given, and within a field of radius 399.0 km., the probable errors for the gravity field are almost negligible compared with those resulting from the neglect of gravity beyond that radius.

Table XVII - Computation of Probable Errors in Gravimetric Deflections as computed for the Gravity Field within Radius 399.0 km.

(1) Zones	(2) No. of Zones	(3) $E_{\Delta g_B}$ (mgal)	(4) E_h (mgal)	(5) E_r (mgal) $\frac{1}{2}[(3)+(4)]$	(6) $0.001\ E_r$ 0.001(5)	(7) $(0.001\ E_r)^2$ $(6)^2$	(8) $\Sigma(0.001E_r)^2$ for Group (2) (7)
LAISKA							
27-30	4			1	0.001	0.000001	0.000004
31-34	4			1	.001	.000001	.000004
35-38	4			1	.001	.000001	.000004
39-42	4			5	.005	.000025	.000100
43-46	4			5	.005	.000025	.000100
47-48	2			5	.005	.000025	.000050
WONDERFONTEIN							
23-26	4	1	1{ 3	2	0.002	0.000004	0.000016
27-30	4	1	3	2	.002	.000004	.000016
31-34	4	1	3	2	.002	.000004	.000016
35-38	4	5	2{ 7	6	.006	.000036	.000144
39-42	4	5	3{ 17	11	.011	.000121	.000484
43-46	4	5	17	11	.011	.000121	.000484
47-48	2	5	17	11	.011	.000121	.000242

1 100 ft. @ 0.03378 mgal / ft.
2 200 ft. @ 0.03378 mgal / ft.
3 500 ft. @ 0.03378 mgal / ft.

Inner Circles:-

	Laiska		Wonderfontein	
ξ : $E_t = d\xi$	+ 0".010		+ 0.551	
E_t^2		0.000100		0.303601
η : $E_t = d\eta$	- 0.268		- 0".105	
E_t^2		0.071824		0.011025
θ^2 : $E_t^2 = d\xi^2 + d\eta^2$		0.071924		0.314626

Table XVII (cont'd)

$E_t^2 = 18\ \Sigma(0.001\ Er)^2$; E_t = p.e. in either component (ξ or η) of the deflection

(9) Zones	(10) E_t^2 18 (8)	(11) Accumulated E_t^2	(12) Accumulated E_t $\sqrt{(11)}$
LAISKA			
27-30	0.000072	0.000072	± 0".008
31-34	.000072	.000144	.012
35-38	.000072	.000216	.015
39-42	.001800	.002016	.045
43-46	.001800	.003816	.062
47-48	.000900	.004716	.068

For Total Deflection θ : $E_t^2 = 2(0.004716)$
$= 0.009432$
$E_t = 0''.097$

WONDERFONTEIN			
23-26	0.000288	0.000288	± 0".017
27-30	.000288	.000576	.024
31-34	.000288	.000864	.029
35-38	.002592	.003456	.059
39-42	.008712	.012168	.110
43-46	.008712	.020880	.145
47-48	.004356	.025236	.159

For Total Deflection θ : $E_t^2 = 2(0.025236)$
$= 0.050472$
$E_t = 0''.225$

[illegible]

[illegible]

[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

Table XVIII - Probable Errors in Gravimetric Deflections θ for Combined Zones

Zones	E_t^2	Accumulated E_t^2	Accumulated E_t
LAISKA			
Inner Circle (r_o = 10.25 km.)	0.0719	0.0719	0".268
27-48 (10.25 - 399.0 km.)	0.0094	0.0813	0".285
Beyond 399.0 km. : $(2''.4)^2$	5.76	5.8413	2".42
WONDERFONTEIN			
Inner Circle (r_o = 5.125 km.)	0.3146	0.3146	0".561
23-26 (5.125 - 399.0 km.)	0.0505	0.3651	0".604
Beyond 399.0 km. : $(2''.4)^2$	5.76	6.1251	2".47

7.5 Approximate Correction to Finland - South Africa Arc from Gravimetric Deflections

Making use of the values obtained for the gravimetric deflections of Laiska and Wonderfontein in section 7.4, disregarding the uncertainty arising from the fact that computation was not carried out beyond 399 km., a correction to the length of the meridian arc between the points can be made. From this, a rough idea of the corresponding correction to the equatorial radius can be learned.

Geodetic and astronomic data for the two points in question was obtained from the Army Map Service, and was part of the data used in that agency's computation of "A New Determination of the Figure of the Earth from Arcs" (See Chovitz and Fischer (4)). Unfortunately the data carries a security classification for values having accuracy better than one minute of arc. Because of this, the actual computation figures cannot be published here. However, the results given reflect the data furnished to the writer by AMS without violating security regulations.

Using latitude functions tables (either (34) or (36)) for the International Ellipsoid, upon which the given data was based, the length of the meridian arc between Laiska and Wonderfontein for the geodetic latitudes and for the latitudes corrected by the free-air gravimetric and astro-geodetic deflections, were computed (See figure 18). In effect, the two figures thus obtained were (1) the measured length of the arc, based on the International Ellipsoid, and (2) the corrected length of the arc based on astro-gravimetric data. The

following figures give pertinent details:-

Station	Latitude		Meridian Arc
Laiska	$\varphi = 64^\circ\ 03'(\ )''$	N	M_L
Wonderfontein	$\varphi = 25^\circ\ 48'(\ )''$	S	M_W
	$\Delta\varphi = 89^\circ\ 51'(\ )''$		

$$(M_L + M_W) = 9{,}961{,}448 \text{ m.}$$

Laiska	$\varphi_g = 64^\circ\ 03'(\ )''$	N	$(M_L)_g$
Wonderfontein	$\varphi_g = 25^\circ\ 48'(\ )''$	S	$(M_W)_g$
	$\Delta\varphi_g = 89^\circ\ 51'(\ )''$		

$$(M_L + M_W)_g = 9{,}961{,}534 \text{ m.}$$

where M_L and M_W refer to the meridian distance from the equator to the point in question, φ is the geodetic latitude, and φ_g the gravimetrically corrected latitude. Thus $\varphi_g = \varphi + (\varphi' - \varphi) - \xi_g$.

From the above figures, we see that the total correction for the arc is + 86 meters. Assuming the arc to be a meridian arc of a sphere the size of the earth, we can thus obtain a rough correction to the sphere's radius:-

$$\delta R = \frac{\rho^\circ}{\Delta\varphi} \times 86$$

$$= \frac{57^\circ\!.3}{89^\circ\!.85} \times 86 = +\ 55 \text{ meters} \approx \delta a$$

Admittedly, this is only a rough approximation, but is close enough to the exact value for the data used, for illustrative purposes. Besides this, the accuracy of the gravimetric deflections is of doubt-

ful value, because of the limited gravity field used.

Any person who is lucky enough to be in possession of data such as is used here, but for many points along an arc of this length, would proceed along the lines of section 7.2, and would doubtless obtain results of some significance.

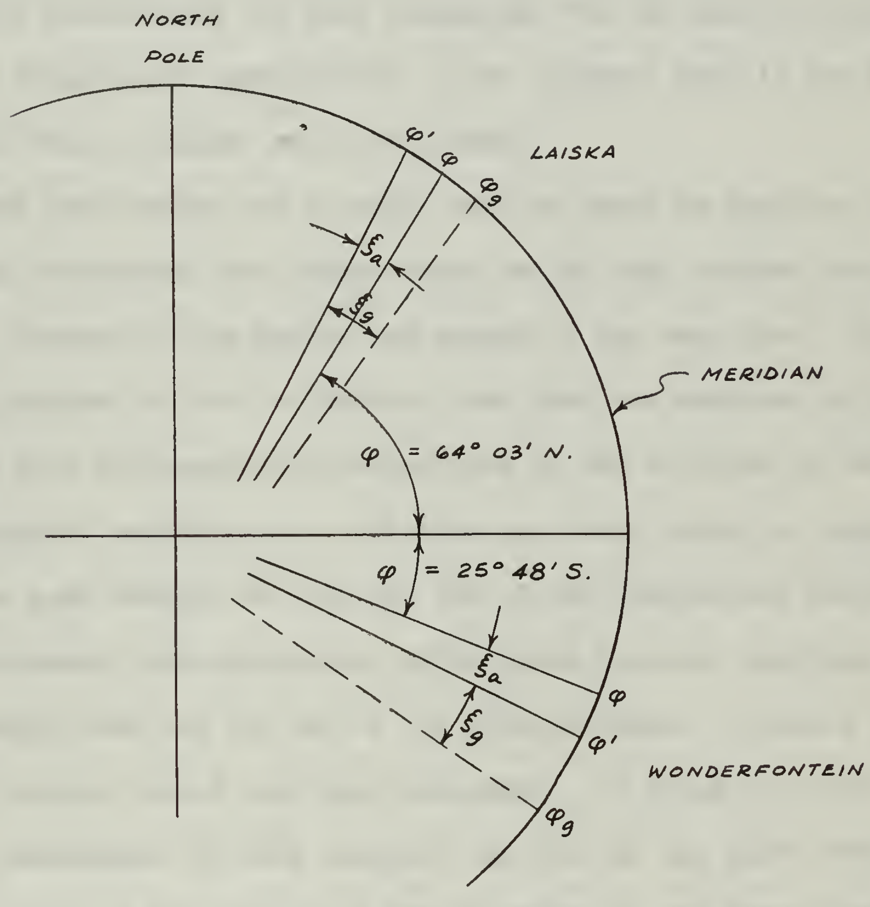


Fig. 18

The Finland - South Africa Arc

Chapter 8 - Closing Remarks

It would be presumptuous on the part of the writer to title this chapter "Conclusions", because there can hardly be any conclusion to something that has not, in a manner of speaking, even begun. The lack of readily available data for pursuing a project of such magnitude as determining the best dimensions for the earth ellipsoid, prevents a significant contribution to the science, even if the ability to put such data to proper use were present.

What conclusions can be made, must be based on what was already known beforehand, and indeed could not be made without such knowledge, because of the nature and amount of the data used. The first two problems solved in Chapter 7 are what are referred to here. The use of only astro-geodetic deflections of the vertical is admittedly an incomplete beginning to a solution, so could hardly be expected to yield as good results as with the use of the gravimetric deflections. It is unfortunate that gravimetric deflections were not available for all the points used for the arc of the 98th meridian. A direct comparison of methods could then have been made. It would have been almost as meaningless to have computed the size of the earth from only the astro-geodetic deflections of the 16 points in the South Central U.S., because of the small area covered. As it was, the computation by the astro-gravimetric method in the case of the 16 points, probably owed much of its "success" to the fact that the data given, as prepared by Mr. Rice, was carefully computed by him and fairly

accurate.

It is hoped that the material presented in this thesis might provide, by its systematic presentation of the procedure to be followed in problems of this nature, the means whereby any interested reader might pursue the problem further, especially when in the future more gravimetric deflections of the vertical can be expected to be available. The advent of the high speed electronic computer in geodetic use will hasten that day. The manual method of computing gravimetric deflections is so laborious that one person, in the work of preparation of a thesis, could not be expected to produce the required computations in adequate numbers and for those stations in locations favorable to solution of the problem of determining the size of the earth ellipsoid. It is believed that the example deflection computations included in this thesis demonstrate this clearly.

It is regretted that more than mere mention of two important methods used for solution of the problem at hand could not have been made in this thesis. One is the determination of the figure of the earth from geoid heights, and the other is the determination of the flattening from observations of the artificial earth satellites. Time, more than space, prevented their inclusion, especially since the writer tends to be somewhat verbose in his explanations.

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